TENSOR CALCULUS with applications to Differential Theory of Surfaces and Dynamics

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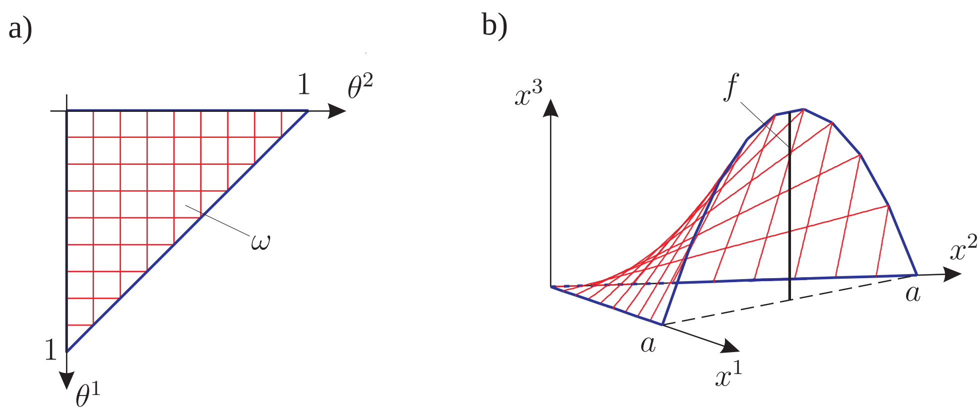
Søren R. K. Nielsen



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Preface

The present outline on tensor calculus with special application to differential theory of surfaces and dynamics represents a modified and extended version of a lecture note written by the author as an introduction to a course on shell theory given together with Ph.D. Jesper Winther Stærdahl and Professor Lars Vabbersgaard Andersen in 2007, based on the book of (Niordson, 1985). The text is written with inspiration from both mathematical based texts on tensor calculus, such as the books of (Spain, 1965) and (Synge and Schild, 1966), and the more geometrical based interpretation often used in continuum mechanics, (Malvern, 1969).

Chapter 1 introduces the concept of vectors and tensors in a Riemann space, and their components in covariant and contravariant vector and tensor bases. Next, the concepts of gradient of vectors and tensors, as well as co- and contravariant derivatives are introduced. Finally, the Riemann tensor and the concept of geodesics curves in Riemann space is treated.

Chapter 2 deals with the differential theory of a surface in the three dimension Euclidean space, as described by its first and second fundamental form. Further, the Bianchi identities for the surface Riemann-Christoffel tensor, and the Codazzi equation for the second fundamental form are derived.

Chapter 3 deals with the description of the motion of a mass particle in curvilinear coordinates and of a non-linear multi-degree-of-freedom dynamic system, which conveniently may be formulated in tensor notation.

Thanks to Ph.D. student Tao Sun for preparing the figures.

Aalborg, May 2018

Søren R.K. Nielsen

CHAPTER 1
Tensor Calculus

1.1 Vectors, curvilinear coordinates, covariant and contravariant bases

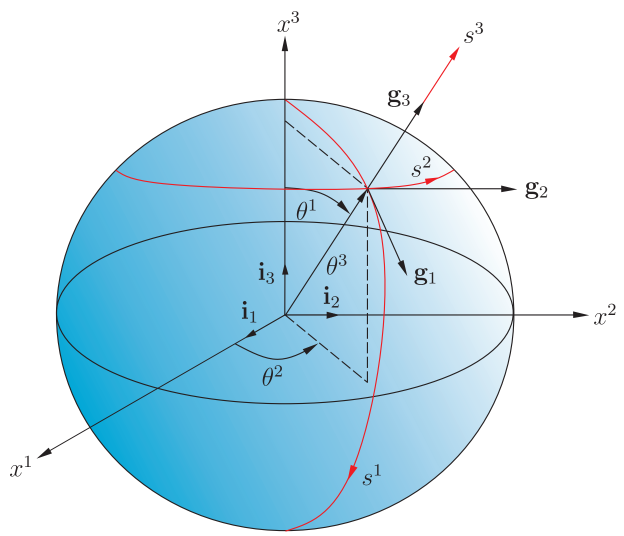

Fig. 1–1 Spherical coordinate system.

Fig. 1-1 shows a *Cartesian* (x^1, x^2, x^3)-*coordinate system*, as well as a *spherical* $(\theta^1, \theta^2, \theta^3)$-coordinate system. θ^1 is the *zenith angle*, θ^2 is the *azimuth angle*, and θ^3 is the *radial distance*. Notice that superscripts are use for the identification of the coordinates, which should not be confused with power raising. Instead, this will be indicated by parentheses, so if x^1 specifies the first Cartesian coordinate, $(x^1)^2$ indicates the corresponding coordinate raised to the second power. With the restrictions $\theta^1 \in [0, \pi]$, $\theta^2 \in]0, 2\pi]$ and $\theta^3 \geq 0$ a one-to-one correspondence between the coordinates of the two systems exists except for points at the line $x^1 = x^2 = 0$. These represent the *singular points* of the mapping. For *regular points* the relations become, see e.g. (Zill and Cullen, 2005):

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \theta^3 \sin\theta^1 \cos\theta^2 \\ \theta^3 \sin\theta^1 \sin\theta^2 \\ \theta^3 \cos\theta^1 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \begin{bmatrix} \cos^{-1} \dfrac{x^3}{\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}} \\ \tan^{-1} \dfrac{x^2}{x^1} \\ \sqrt{(x^1)^2+(x^2)^2+(x^3)^2} \end{bmatrix} \tag{1–1}$$

We shall refer to θ^l, $l = 1, 2, 3$, as *curvilinear coordinates*. Generally, the relation between the Cartesian and curvilinear coordinates are given by relations of the type:

$$x^j = f^j(\theta^l) \tag{1–2}$$

The *Jacobian* of the mapping (1-2) is defined as:

$$J = \det \left[\frac{\partial f^j}{\partial \theta^k} \right] \tag{1–3}$$

Points, where $J = 0$, represent *singular points* of the mapping. In any regular point, where $J \neq 0$, the inverse mapping exists locally, given as:

$$\theta^j = h^j(x^l) \tag{1–4}$$

For $\theta^1 = c_1$ (constant), (1-2) defines a surface in space, defined by the parametric description:

$$x^j = f^j(c^1, \theta^2, \theta^3) \tag{1–5}$$

Similar parametric description of surfaces arise, when θ^2 or θ^3 are kept constant, and the remaining two coordinates are varied independently. The indicated three surfaces intersect pairwise along three curves s^j, $j = 1, 2, 3$, at which two of the curvilinear coordinates are constant, e.g. the intersection curve s^1 is determined from the parametric description $x^j = f^j(\theta^1, c^2, c^3)$. All three surfaces intersect at the point P with the Cartesian coordinates $x^j = f^j(c^1, c^2, c^3)$. Locally, at this point an additional curvilinear (s^1, s^2, s^3)-coordinate system may be defined with axes made up of the said intersection curves as shown on Fig. 1-1. We shall refer to these coordinates as the *arc length coordinates*.

The position vector $\mathbf{x}$ from the origin of the Cartesian coordinate system to the point P has the vector representation:

$$\mathbf{x} = x^1\mathbf{i}_1 + x^2\mathbf{i}_2 + x^3\mathbf{i}_3 = x^j\mathbf{i}_j \tag{1–6}$$

where $\mathbf{i}_j$, $j = 1, 2, 3$, signify the orthonormal *base vectors* of the Cartesian coordinate system. In the last statement the *summation convention* over dummy indices has been used. This convention will extensively be used in what follows. The rules is that dummy Latin indices involves summation over the range $j = 1, 2, 3$, whereas dummy Greek indices merely involves summation over the range $\alpha = 1, 2$. As an example $a^j b_j = a^1 b_1 + a^2 b_2 + a^3 b_3$, whereas $a^\alpha b_\alpha = a^1 b_1 + a^2 b_2$. The summation convention is abolished, if the dummy indices are surrounded by parentheses, i.e. $a^{(j)} b_{(j)}$ merely means the product of the jth components a^j and b_j.

Let dx^j denote an infinitesimal increment of the jth coordinate, whereas the other coordinates are kept constant. From (1-6) follows that this induces a change of the position vector given as $d\mathbf{x} = \mathbf{i}_{(j)} dx^{(j)}$. Hence:

$$\mathbf{i}_j = \frac{\partial \mathbf{x}}{\partial x^j} \tag{1–7}$$

A corresponding independent infinitisimal increment $d\theta^j$ of the jth curvilinear coordinate induce a change of the position vector given as $d\mathbf{x} = \mathbf{g}_{(j)} d\theta^{(j)}$, so:

$$\mathbf{g}_j = \frac{\partial \mathbf{x}}{\partial \theta^j} \tag{1–8}$$

$\mathbf{g}_j$ is tangential to the curve s^j at the point P, see Fig. 1-1. In any regular point the vectors $\mathbf{g}_j$, $j = 1, 2, 3$, may be used as base vector for the arc length coordinate system at P. The indicated vector base vectors is referred to as the *covariant vector base*, and $\mathbf{g}_j$ are denoted the *covariant base vectors*. Especially, if the motion is described in arc length coordinates, $\mathbf{g}_j$ becomes equal to the *unit tangent vectors* $\mathbf{t}_j$, i.e.:

$$\mathbf{t}_j = \frac{\partial \mathbf{x}}{\partial s^j} \tag{1–9}$$

Use of the chain rule of partial differentiation provides:

$$\mathbf{g}_j = \frac{\partial \mathbf{x}}{\partial \theta^j} = \frac{\partial \mathbf{x}}{\partial x^k} \frac{\partial x^k}{\partial \theta^j} = c_j^k \, \mathbf{i}_k \tag{1–10}$$

$$\mathbf{i}_j = \frac{\partial \mathbf{x}}{\partial x^j} = \frac{\partial \mathbf{x}}{\partial \theta^k} \frac{\partial \theta^k}{\partial x^j} = d_j^k \, \mathbf{g}_k \tag{1–11}$$

where:

$$\left. \begin{aligned} c_j^k &= \frac{\partial x^k}{\partial \theta^j} = \frac{\partial f^k}{\partial \theta^j} \\ d_j^k &= \frac{\partial \theta^k}{\partial x^j} = \frac{\partial h^k}{\partial x^j} \end{aligned} \right\} \tag{1–12}$$

c_j^k specify the Cartesian components of the covariant base vector $\mathbf{g}_j$. Similarly, d_j^k specifies the components in the covariant base of the Cartesian base vector $\mathbf{i}_j$. Obviously, the following relation is valid:

$$c^k_l \, d^l_j \; = \; \frac{\partial x^k}{\partial \theta^l} \frac{\partial \theta^l}{\partial x^j} = \delta^k_j \tag{1–13}$$

δ^k_j denotes *Kronecker's delta* in the applied notation, defined as:

$$\delta^k_j \; = \; \begin{cases} 0 & , \quad j \neq k \\ 1 & , \quad j = k \end{cases} \tag{1–14}$$

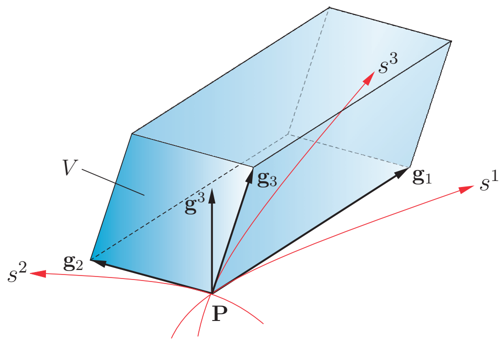

Fig. 1–2 Covariant and contravariant base vectors.

Generally, the covariant base vectors $\mathbf{g}_j$ are neither orthogonal nor normalized to the length 1, as is the case for the Cartesian base vectors $\mathbf{i}_j$. In order to perform similar operations on components of vectors and tensors as for an orthonormal vector base, a so-called *contravariant vector base* or *dual vector base* is introduced. The corresponding *contravariant base vectors* $\mathbf{g}^j$ are defined from:

$$\mathbf{g}_j \cdot \mathbf{g}^k \; = \; \delta^k_j \tag{1–15}$$

where "·" indicates a scalar product. (1-15) implies that $\mathbf{g}^3$ is orthogonal to $\mathbf{g}_1$ and $\mathbf{g}_2$. Further, the angle between $\mathbf{g}_3$ and $\mathbf{g}^3$ is acute in order that $\mathbf{g}_3 \cdot \mathbf{g}^3 = +1$, see Fig. 1-2. Generally, the contravariant base vectors can be determined from the covariant base vectors by means of the vector products:

$$\left.\begin{aligned} \mathbf{g}^1 &= \frac{1}{V}\, \mathbf{g}_2 \times \mathbf{g}_3 \\ \mathbf{g}^2 &= \frac{1}{V}\, \mathbf{g}_3 \times \mathbf{g}_1 \\ \mathbf{g}^3 &= \frac{1}{V}\, \mathbf{g}_1 \times \mathbf{g}_2 \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{g}^i = \frac{1}{V}\, e^{ijk}\, \mathbf{g}_j \times \mathbf{g}_k \tag{1–16}$$

where V denotes the volume of the parallelepiped spanned by the covariant base vectors $\mathbf{g}_j$, see Fig. 1-2. This is given as:

$$V = \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) = \mathbf{g}_2 \cdot (\mathbf{g}_3 \times \mathbf{g}_1) = \mathbf{g}_3 \cdot (\mathbf{g}_1 \times \mathbf{g}_2) \tag{1–17}$$

e^{ijk} is the *permutation symbol* defined as:

$$e^{ijk} = \begin{cases} 1 & , \quad (i,j,k) = (1,2,3)\,,\,(2,3,1)\,,\,(3,1,2) \\ -1 & , \quad (i,j,k) = (1,3,2)\,,\,(3,2,1)\,,\,(2,1,3) \\ 0 & , \quad \text{else} \end{cases} \tag{1–18}$$

The permutation symbol does not indicate the components of a 3rd order tensor in any coordinate system, and should merely be considered as an indexed sequence of numbers. For this reason we shall not make distinction between subscript and superscript indices, so we may write $e_{ijk} = e^{ijk}$. The permutation symbol and the Kronecker delta are related by the following so-called $e - \delta$ *relation*:

$$e^{ijk} e_{klm} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l \tag{1–19}$$

In the Cartesian coordinate system we have $\mathbf{i}^j = \mathbf{i}_j$, i.e. the dual vector basis is identical to original. Further, the Cartesian base vectors are constant throughout the space, whereas the covariant and contravariant base vector are locally attached to each point in the space, and change from point to point.

The previous theory merely applies to a 3-dimensional Euclidean spaces. In the following this is generalized to a *Riemann space* of arbitrary dimension N. A Riemann space is a manifold related with a distance measure, which is defined between any two points in the space. A curve and surface in the 3-dimensional Euclidian space are examples of Riemann spaces of dimension $N = 1$ and $N = 2$. The space-time manifold in relativity theory is a Riemann space of dimension $N = 4$.

Similar to the concept in the 2- and 3-dimensional Euclidian spaces a vector in Riemann space is envisioned as a geometric quantity (an ”arrow” with a given length and orientation). From this interpretation it follows that a vector is independent of any coordinate system used for its specification. Actually, infinitely many coordinate systems can be used for the representation (or decomposition) of one and the same vector. In the Cartesian, the covariant and the contravariant bases a given vector $\mathbf{v}$ can be represented in the following ways:

$$\mathbf{v} = \bar{v}^j \, \mathbf{i}_j = v^j \, \mathbf{g}_j = v_j \, \mathbf{g}^j \tag{1–20}$$

where $\bar{v}^j$, v^j and v_j denotes the *Cartesian vector components*, the *covariant vector components*, and the *contravariant vector components* of the vector $\mathbf{v}$. Dummy indices now indicates summation over the range $1, \ldots, N$. Generally, Cartesian components of vectors and tensors will be indicated by a bar. Use of (1-15), and scalar multiplication of the two last relations in (1-20) with $\mathbf{g}_k$ and $\mathbf{g}^k$, respectively, provides the following relations between the covariant and contravariant vector components:

$$\left. \begin{aligned} v_j &= g_{jk} v^k \\ v^j &= g^{jk} v_k \end{aligned} \right\} \tag{1–21}$$

where:

$$\left. \begin{aligned} g_{jk} &= \mathbf{g}_j \cdot \mathbf{g}_k = g_{kj} \\ g^{jk} &= \mathbf{g}^j \cdot \mathbf{g}^k = g^{kj} \end{aligned} \right\} \tag{1–22}$$

The indicated symmetry property of the quantities g_{jk} and g^{jk} follows from the commutativity of the involved scalar products. From (1-21) follows:

$$v_j = g_{jl} v^l = g_{jl} g^{lk} v_k \quad \Rightarrow \quad g_{jl} g^{lk} = \delta_j^k \tag{1–23}$$

By the use of (1-20), (1-21) and (1-22) the following relations between the covariant and the contravariant base vectors may be derived:

$$\left. \begin{aligned} v^j \, \mathbf{g}_j &= v_j \, \mathbf{g}^j = g_{jk} v^k \, \mathbf{g}^j = v^j g_{jk} \, \mathbf{g}^k \\ v_j \, \mathbf{g}^j &= v^j \, \mathbf{g}_j = g^{jk} v_k \, \mathbf{g}_j = v_j g^{jk} \, \mathbf{g}_k \end{aligned} \right\} \quad \Rightarrow \quad \left\{ \begin{aligned} \mathbf{g}_j &= g_{jk} \, \mathbf{g}^k \\ \mathbf{g}^j &= g^{jk} \, \mathbf{g}_k \end{aligned} \right. \tag{1–24}$$

As seen g_{jk} represent the components of $\mathbf{g}_j$ in the contravariant vector base. Similarly g^{jk} signify the components of $\mathbf{g}^j$ in the covariant vector base. Use of (1-10) and (1-11) provides the following relation between the Cartesian and the covariant vector components:

$$\left. \begin{aligned} \bar{v}^j \, \mathbf{i}_j &= v^j \, \mathbf{g}_j = v^j c_j^k \, \mathbf{i}_k = c_k^j v^k \, \mathbf{i}_j \\ v^j \, \mathbf{g}_j &= \bar{v}^j \, \mathbf{i}_j = \bar{v}^j d_j^k \, \mathbf{g}_k = d_k^j \bar{v}^k \, \mathbf{g}_j \end{aligned} \right\} \quad \Rightarrow \quad \left\{ \begin{aligned} \bar{v}^j &= c_k^j v^k \\ v^j &= d_k^j \bar{v}^k \end{aligned} \right. \tag{1–25}$$

Finally, using (1-15) and (1-21) the scalar product of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be evaluated in the following alternative ways:

$$\mathbf{u} \cdot \mathbf{v} = \left\{ \begin{aligned} \bar{u}^j \bar{v}_j &= u^j v_j = g^{jk} u_j v_k \\ \bar{u}_j \bar{v}^j &= u_j v^j = g_{jk} u^j v^k \end{aligned} \right. \tag{1–26}$$

where it has been used that $\bar{u}^j = \bar{u}_j$.

1.2 Tensors, dyads and polyads

A *second order tensor* $\mathbf{T}$ is defined as a linear mapping of a vector $\mathbf{v}$ onto a vector $\mathbf{u}$ by means of a scalar product, i.e.:

$$\mathbf{u} = \mathbf{T} \cdot \mathbf{v} \tag{1–27}$$

Since the vectors $\mathbf{u}$ and $\mathbf{v}$ are coordinate independent quantities, the 2nd order tensor $\mathbf{T}$ must also be independent of any selected coordinate system chosen for the specification of the relation (1-27). Equations in continuum mechanics and physics are independent of the chosen coordinate system for which reason these are basically formulated as tensor equations.

A *dyad* (or *outer product* or *tensor product*) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted as $\mathbf{ab}$. An alternatively often applied notation, which will not be used here, is $\mathbf{a} \otimes \mathbf{b}$. The tensor product of more than two vectors is denoted a *polyad*. The polyad $\mathbf{abc}$ formed by the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is denoted a *triad*, and the polyad $\mathbf{abcd}$ formed by the four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ is denoted a *tetrad*.

For dyads and triads the following *associative rules* apply:

$$\left.\begin{aligned} m\,(\mathbf{ab}) \;&=\; (m\mathbf{a})\,\mathbf{b} \;=\; \mathbf{a}\,(m\mathbf{b}) \;=\; (\mathbf{ab})\,m \\ \mathbf{abc} \;&=\; (\mathbf{ab})\,\mathbf{c} \;=\; \mathbf{a}\,(\mathbf{bc}) \\ (m\mathbf{a})\,(n\mathbf{b}) \;&=\; mn\,\mathbf{ab} \end{aligned}\right\} \tag{1–28}$$

where m and n are arbitrary constants. Further, the following *distributive rules* are valid:

$$\left.\begin{aligned} \mathbf{a}\,(\mathbf{b}+\mathbf{c}) \;&=\; \mathbf{ab}+\mathbf{ac} \\ (\mathbf{a}+\mathbf{b})\,\mathbf{c} \;&=\; \mathbf{ab}+\mathbf{bc} \end{aligned}\right\} \tag{1–29}$$

No commutative rule is valid for dyads formed by two vectors $\mathbf{a}$ and $\mathbf{b}$. Hence, in general:

$$\mathbf{ab} \neq \mathbf{ba} \tag{1–30}$$

If the outer product of two vectors entering a polyad is replaced by a scalar product of the same vectors, a polyad of an order two smaller is obtained. This operation is known as *contraction*. Contraction of a triad is possible in the following two ways:

$$\left.\begin{aligned} \mathbf{a}\cdot\mathbf{bc} \;&=\; (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \\ \mathbf{ab}\cdot\mathbf{c} \;&=\; (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} \end{aligned}\right\} \tag{1–31}$$

The scalar product of two dyads, the so-called *double contraction*, can be defined in two ways:

$$\left.\begin{aligned} \mathbf{ab}:\mathbf{cd} \;&=\; (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \\ \mathbf{ab}\cdot\cdot\,\mathbf{cd} \;&=\; (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \end{aligned}\right\} \tag{1–32}$$

Hence, the symbol " : " defines scalar multiplication between the two first and the two last vectors in the two duads, whereas " $\cdot\cdot$ " defines scalar multiplication between the first and the last vector, and the last and the first vectors in the two dyads. Because of the commutativity of the scalar product of two vectors follows:

$$\left.\begin{aligned} (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \;&=\; (\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{c}) \;=\; (\mathbf{c}\cdot\mathbf{a})(\mathbf{d}\cdot\mathbf{b}) \;=\; (\mathbf{d}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{a}) \\ (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \;&=\; (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) \;=\; (\mathbf{c}\cdot\mathbf{b})(\mathbf{d}\cdot\mathbf{a}) \;=\; (\mathbf{d}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{b}) \end{aligned}\right\} \tag{1–33}$$

Use of (1-32) and (1-33) provides the following identities for double contraction of two dyads:

$$\left.\begin{aligned} \mathbf{ab}:\mathbf{cd} &= \mathbf{ba}:\mathbf{dc} = \mathbf{cd}:\mathbf{ab} = \mathbf{dc}:\mathbf{ba} \\ \mathbf{ab}\cdot\cdot\,\mathbf{cd} &= \mathbf{ba}\cdot\cdot\,\mathbf{dc} = \mathbf{cd}\cdot\cdot\,\mathbf{ab} = \mathbf{dc}\cdot\cdot\,\mathbf{ba} \end{aligned}\right\} \tag{1–34}$$

From the second relation of (1-31) follows that the dyad $\mathbf{ab}/(\mathbf{b}\cdot\mathbf{c})$ is mapping the vector $\mathbf{c}$ onto the vector $\mathbf{a}$ via a scalar multiplication. From the definition (1-27) follows that this makes a dyad a second order tensor. Now, it can be proved that any second order tensor can be represented as a linear combination of nine dyads, formed as outer product of three arbitrary linearly independent base vectors. Hence, we have following alternative representations of a second order tensor $\mathbf{T}$ in the Cartesian, a covariant base and its contravariant base:

$$\mathbf{T} = \bar{T}^{jk}\,\mathbf{i}_j\mathbf{i}_k = T^{jk}\,\mathbf{g}_j\mathbf{g}_k = T_{jk}\,\mathbf{g}^j\mathbf{g}^k \tag{1–35}$$

The dyads $\mathbf{i}_j\mathbf{i}_k$, $\mathbf{g}_j\mathbf{g}_k$ and $\mathbf{g}^j\mathbf{g}^k$ form so-called *tensor bases*. Obviously, (1-35) represents the generalization to second order tensors of (1-20) for the decomposition of a vector in the corresponding vector bases. $\bar{T}^{jk}$, T^{jk} and T_{jk} denotes the *Cartesian components*, the *covariant components*, and the *contravariant components* of the second order tensor $\mathbf{T}$. Using (1-10), (1-11), (1-13) and the associate rules (1-28) the following relations between the dyads and tensor components related to the considered three tensor bases may be derived:

$$\left.\begin{aligned} \mathbf{i}_j\mathbf{i}_k &= d^l_j d^m_k\,\mathbf{g}_l\mathbf{g}_m \\ \mathbf{g}_j\mathbf{g}_k &= c^l_j c^m_k\,\mathbf{i}_l\mathbf{i}_m = g_{jl}g_{km}\,\mathbf{g}^l\mathbf{g}^m \\ \mathbf{g}^j\mathbf{g}^k &= g^{jl}g^{km}\,\mathbf{g}_l\mathbf{g}_m \end{aligned}\right\} \tag{1–36}$$

$$\left.\begin{aligned} \bar{T}^{jk} &= c^j_l c^k_m T^{lm} \quad , \quad T^{jk} = d^j_l d^k_m \bar{T}^{lm} \\ T_{jk} &= g_{jl}g_{km}T^{lm} \quad , \quad T^{jk} = g^{jl}g^{km}T_{lm} \end{aligned}\right\} \tag{1–37}$$

Hence, $c^l_j c^m_k$ and $g_{jl}g_{km}$ specify the Cartesian and contravariant tensor components of the dyad $\mathbf{g}_j\mathbf{g}_k$, whereas $g^{jl}g^{km}$ denotes the covariant tensor components of $\mathbf{g}^j\mathbf{g}^k$. (1-36) and (1-37) represent the equivalent to the relations (1-23) and (1-24) for base vectors and vector components. In some outlines of tensor calculus the transformation rules in (1-37) between Cartesian and curvilinear tensor components are used as a defining property of the tensorial character of a doubled indexed quantity, (Spain, 1965), (Synge and Schild, 1966).

Alternatively, $\mathbf{T}$ may be decomposed after a tensor base with dyads of mixed covariant and contravariant base vectors, i.e.:

$$\mathbf{T} = T^j{}_k\,\mathbf{g}_j\mathbf{g}^k = T_k{}^j\,\mathbf{g}^k\mathbf{g}_j \tag{1–38}$$

$T^j{}_k$ and $T_k{}^j$ represent the *mixed covariant and contravariant tensor components*. In general, $T^j{}_k \neq T_k{}^j$ as a consequence of $\mathbf{g}_j\mathbf{g}^k \neq \mathbf{g}^k\mathbf{g}_j$, i.e. the relative horizontal position of the upper and lower indices of the tensor components is important, and specify the sequence of covariant and contravariant base vector in the dyads of the tensor base. Tensor bases with dyads of mixed Cartesian and curvilinear base vectors may also be introduced. However, in what follows only the mixed curvilinear dyads in (1-38) will be considered. The following identities may be derived from (1-35) and (1-38) by the use of (1-24):

$$\left.\begin{aligned}\mathbf{T} = T^{jk}\,\mathbf{g}_j\mathbf{g}_k &= T^j{}_l\,\mathbf{g}_j\mathbf{g}^l &&= T^j{}_l g^{lk}\,\mathbf{g}_j\mathbf{g}_k\\ \mathbf{T} = T^{jk}\,\mathbf{g}_j\mathbf{g}_k &= T_l{}^k\,\mathbf{g}^l\mathbf{g}_k &&= T_l{}^k g^{lj}\,\mathbf{g}_j\mathbf{g}_k\\ \mathbf{T} = T_{jk}\,\mathbf{g}^j\mathbf{g}^k &= T^l{}_k\,\mathbf{g}_l\mathbf{g}^k &&= T^l{}_k g_{lj}\,\mathbf{g}^j\mathbf{g}^k\\ \mathbf{T} = T_{jk}\,\mathbf{g}^j\mathbf{g}^k &= T_j{}^l\,\mathbf{g}^j\mathbf{g}_l &&= T_j{}^l g_{lk}\,\mathbf{g}^j\mathbf{g}^k\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}T^{jk} &= g^{lk}\,T^j{}_l\\ T^{jk} &= g^{jl}\,T_l{}^k\\ T_{jk} &= g_{jl}\,T^l{}_k\\ T_{jk} &= g_{lk}\,T_j{}^l\end{aligned}\right. \tag{1–39}$$

Use of (1-23) and (1-39) provides the following representations of the mixed components in terms of covariant and contravariant tensor components:

$$\left.\begin{aligned}T^k{}_j &= g_{jl}\,T^{kl}\\ T_j{}^k &= g_{jl}\,T^{lk}\\ T^k{}_j &= g^{kl}\,T_{lj}\\ T_j{}^k &= g^{kl}\,T_{jl}\end{aligned}\right\} \tag{1–40}$$

It follows from (1-40) that if $T^{jk} = T^{kj}$ then $T^k_{\ j} = T_k^{\ j}$ and $T_{jk} = T_{kj}$. A second order tensor for which the covariant components fulfill the indicated symmetry property is denoted a *symmetric tensor*.

Let T^{jk} denote the covariant components of a second order tensor $\mathbf{T}$. The related so-called *transposed tensor* $\mathbf{T}^T$ is defined as the tensor with the covariant components T^{kj}, i.e.:

$$\mathbf{T}^T = T^{kj}\,\mathbf{g}_j\mathbf{g}_k \tag{1–41}$$

A *symmetric tensor* is defined by the symmetry of components $T^{jk} = T^{kj}$ for which reason $\mathbf{T}^T = \mathbf{T}$.

The *identity tensor* is defined as the tensor with the covariant and contravariant components g^{jk} and g_{jk}. Use of (1-23) and (1-24) provides:

$$\mathbf{g} = g^{jk}\,\mathbf{g}_j\mathbf{g}_k = g_{jk}\,\mathbf{g}^j\mathbf{g}^k = \delta^k_j\,\mathbf{g}^j\mathbf{g}_k \tag{1–42}$$

The mixed components of g follows from (1-23) and (1-40):

$$\left.\begin{aligned}g^k{}_j &= g_{jl}\,g^{kl} = \delta^k_j\\ g_j{}^k &= g_{jl}\,g^{lk} = \delta^k_j\end{aligned}\right\} \tag{1–43}$$

Hence, the mixed components are equal to the Kronecker's delta, which explains the last statement for the mixed representation in (1-42). The designation identity tensor stems from the fact that g maps any vector $\mathbf{v}$ onto itself. Actually:

$$\mathbf{g} \cdot \mathbf{v} \; = \; g^{jk}\, \mathbf{g}_j \mathbf{g}_k \; \cdot \; v_l\, \mathbf{g}^l \; = \; g^{jk} v_l\, \mathbf{g}_j\, \delta^l_k = g^{jk} v_k\, \mathbf{g}_j \; = \; v^j\, \mathbf{g}_j \; = \; \mathbf{v} \tag{1–44}$$

The length of a vector $\mathbf{v}$ is determined from:

$$|\mathbf{v}|^2 \; = \; \mathbf{v} \, \cdot \, \mathbf{v} \; = \; \mathbf{v} \, \cdot \, \mathbf{g} \, \cdot \, \mathbf{v} \; = \; g^{jk} v_j v_k \; = \; g_{jk} v^j v^k \tag{1–45}$$

Because of its relationship to the length of a vector the identity tensor is also designated the *metric tensor* or the *fundamental tensor*. For an ordinary Riemann space $\mathbf{g}$ is positive definite, so $|\mathbf{v}|^2$ is always positive. In relativity theory the metric tensor is indefinite, so the left hand side in Eq. (1-45) may be negative. A manifold related with an indefinite metric tensor is referred to as a *pseudo-Riemann space*.

The increment $d\mathbf{x}$ of the position vector $\mathbf{x}$ with Cartesian components $d\bar{x}^j$ and covariant components $d\theta^j$ is given as, cf. (1-25):

$$d\mathbf{x} \; = \; d\bar{x}^j \, \mathbf{i}_j \; = \; d\theta^j \, \mathbf{g}_j \tag{1–46}$$

Then, the length $ds = \sqrt{d\mathbf{x} \cdot d\mathbf{x}}$ of the incremental position vector becomes, cf. (1-45):

$$ds^2 \; = \; d\bar{x}^j d\bar{x}_j \; = \; g_{jk} d\theta^j d\theta^k \tag{1–47}$$

The *inverse tensor* $\mathbf{T}^{-1}$ related to the tensor $\mathbf{T}$ is defined from the equation:

$$\mathbf{T}^{-1} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{T}^{-1} = \mathbf{g} \tag{1–48}$$

Let T^{-1}_{jl} denote the contravariant components of $\mathbf{T}^{-1}$, and T^{mk} the covariant components of $\mathbf{T}$. Then, cf. (1-42):

$$\begin{aligned} &\mathbf{g} = \delta^k_j \, \mathbf{g}^j \mathbf{g}_k = \mathbf{T}^{-1} \cdot \mathbf{T} = T^{-1}_{jl} \, \mathbf{g}^j \mathbf{g}^l \; \cdot \; T^{mk} \, \mathbf{g}_m \mathbf{g}_k = T^{-1}_{jl} T^{mk} \delta^l_m \, \mathbf{g}^j \mathbf{g}_k = T^{-1}_{jl} T^{lk} \, \mathbf{g}^j \mathbf{g}_k \quad \Rightarrow \\ &T^{-1}_{jl} T^{lk} = \delta^k_j \end{aligned} \tag{1–49}$$

In matrix notation this means that the N-dimensional matrix, which stores the components T^{-1}_{jl}, is the inverse of the matrix, which stores the components T^{lk}. Similarly, the covariant components $(T^{-1})^{jl}$ of $\mathbf{T}^{-1}$ and the contravariant components T_{lk} of $\mathbf{T}$ are stored in inverse matrices. In contrast, the matrices which store the contravariant components T^{-1}_{jl} and T_{lk} will not be mutual inverse. From (1-15) follows that:

$$\begin{aligned} &\mathbf{g}^j \, \cdot \, \mathbf{T} \, \cdot \, \mathbf{g}^k \; = \; \mathbf{g}^j \, \cdot \, \big(T^{lm} \, \mathbf{g}_l \mathbf{g}_m \big) \, \cdot \, \mathbf{g}^k \; = \; T^{lm} \, \delta^j_l \, \delta^k_m \; = \; T^{jk} \quad \Rightarrow \\ &T^{jk} \; = \; \mathbf{g}^j \, \cdot \, \mathbf{T} \, \cdot \, \mathbf{g}^k \end{aligned} \tag{1–50}$$

Similarly:

$$\left.\begin{aligned} T_{jk} &= \mathbf{g}_j \cdot \mathbf{T} \cdot \mathbf{g}_k \\ T_j^{\ k} &= \mathbf{g}_j \cdot \mathbf{T} \cdot \mathbf{g}^k \\ T^j_{\ k} &= \mathbf{g}^j \cdot \mathbf{T} \cdot \mathbf{g}_k \\ \bar{T}_{jk} &= \mathbf{i}_j \cdot \mathbf{T} \cdot \mathbf{i}_k \end{aligned}\right\} \tag{1–51}$$

A *fourth order tensor* $\mathbf{C}$ can be expanded in any tensor base with tetrads made up of any combination of linear independent vectors of the Cartesian, the covariant or the contravariant vector bases as follows:

$$\left.\begin{aligned} \mathbf{C} &= C^{jklm}\,\mathbf{g}_j\mathbf{g}_k\mathbf{g}_l\mathbf{g}_m = C_j^{\ klm}\,\mathbf{g}^j\mathbf{g}_k\mathbf{g}_l\mathbf{g}_m = C^j_{\ k}{}^{lm}\,\mathbf{g}_j\mathbf{g}^k\mathbf{g}_l\mathbf{g}_m = C^{jk}{}_l{}^m\,\mathbf{g}_j\mathbf{g}_k\mathbf{g}^l\mathbf{g}_m \\ &= C^{jkl}{}_m\,\mathbf{g}_j\mathbf{g}_k\mathbf{g}_l\mathbf{g}^m = C_{jk}{}^{lm}\,\mathbf{g}^j\mathbf{g}^k\mathbf{g}_l\mathbf{g}_m = C_j{}^k{}_l{}^m\,\mathbf{g}^j\mathbf{g}_k\mathbf{g}^l\mathbf{g}_m = C_j{}^{kl}{}_m\,\mathbf{g}^j\mathbf{g}_k\mathbf{g}_l\mathbf{g}^m \\ &= C^j{}_{kl}{}^m\,\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l\mathbf{g}_m = C^j{}_k{}^l{}_m\,\mathbf{g}_j\mathbf{g}^k\mathbf{g}_l\mathbf{g}^m = C^{jk}{}_{lm}\,\mathbf{g}_j\mathbf{g}_k\mathbf{g}^l\mathbf{g}^m = C^j{}_{klm}\,\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l\mathbf{g}^m \\ &= C_j{}^k{}_{lm}\,\mathbf{g}^j\mathbf{g}_k\mathbf{g}^l\mathbf{g}^m = C_{jk}{}^l{}_m\,\mathbf{g}^j\mathbf{g}^k\mathbf{g}_l\mathbf{g}^m = C_{jkl}{}^m\,\mathbf{g}^j\mathbf{g}^k\mathbf{g}^l\mathbf{g}_m = C_{jklm}\,\mathbf{g}^j\mathbf{g}^k\mathbf{g}^l\mathbf{g}^m \\ &= \bar{C}^{jklm}\,\mathbf{i}_j\mathbf{i}_k\mathbf{i}_l\mathbf{i}_m \end{aligned}\right\} \tag{1–52}$$

Formally, the relations between the various mixed tensor components can be derived by raising and lowering indices by means of the covariant components g^{jk} and the contravariant components g_{jk} of the identity tensor g, cf. (1-40). As an example:

$$C^{jk}{}_{lm} = g_{lr} g_{ms}\, C^{jkrs} \tag{1–53}$$

1.3 Gradient, covariant and contravariant derivatives

Consider a scalar function $a = a(x^l) = a(\theta^l)$ of the Cartesian or curvilinear coordinates. The *gradient* of a, denoted as $\boldsymbol{\nabla} a$, is a vector with the following Cartesian and contravariant representations:

$$\boldsymbol{\nabla} a = \frac{\partial a}{\partial x^j}\,\mathbf{i}_j = \frac{\partial a}{\partial \theta^j}\,\mathbf{g}^j \tag{1–54}$$

Let $\Delta\theta^k$ denote the covariant components of the incremental curvilinear coordinate vector $\Delta\boldsymbol{\theta} = \Delta\theta^k\,\mathbf{g}_k$. The corresponding increment Δa of the scalar a is determined by:

$$\Delta a = \boldsymbol{\nabla} a \cdot \Delta\boldsymbol{\theta} = \frac{\partial a}{\partial \theta^j}\,\mathbf{g}^j \cdot \Delta\theta^k\,\mathbf{g}_k = \frac{\partial a}{\partial \theta^j}\,\Delta\theta^j \tag{1–55}$$

The *gradient of a vector function* $\mathbf{v} = \mathbf{v}(x^l) = \mathbf{v}(\theta^l)$, denoted as $\boldsymbol{\nabla}\mathbf{v}$, is a second order tensor, which in complete analogy to (1-55) associates to any incremental curvilinear coordinate vector

$\Delta\boldsymbol{\theta} = \Delta\theta^k\,\mathbf{g}_k$ a corresponding increment $\Delta\mathbf{v} = \boldsymbol{\nabla}\mathbf{v}\cdot\Delta\boldsymbol{\theta} = \frac{\partial\mathbf{v}}{\partial\theta^j}\,\Delta\theta^j$ of the vector $\mathbf{v}$. $\boldsymbol{\nabla}\mathbf{v}$ has the following representations:

$$
\boldsymbol{\nabla}\mathbf{v} = \begin{cases} \dfrac{\partial\mathbf{v}}{\partial x^k}\,\mathbf{i}_k = \dfrac{\partial(\bar{v}^j\,\mathbf{i}_j)}{\partial x^k}\,\mathbf{i}_k = \dfrac{\partial\bar{v}^j}{\partial x^k}\,\mathbf{i}_j\mathbf{i}_k \\[2mm] \dfrac{\partial\mathbf{v}}{\partial\theta^k}\,\mathbf{g}^k = \dfrac{\partial(v^j\,\mathbf{g}_j)}{\partial\theta^k}\,\mathbf{g}^k = \left(\dfrac{\partial v^j}{\partial\theta^k}\,\mathbf{g}_j + v^j\,\dfrac{\partial\mathbf{g}_j}{\partial\theta^k}\right)\mathbf{g}^k \\[2mm] \dfrac{\partial\mathbf{v}}{\partial\theta^k}\,\mathbf{g}^k = \dfrac{\partial(v_j\,\mathbf{g}^j)}{\partial\theta^k}\,\mathbf{g}^k = \left(\dfrac{\partial v_j}{\partial\theta^k}\,\mathbf{g}^j + v_j\,\dfrac{\partial\mathbf{g}^j}{\partial\theta^k}\right)\mathbf{g}^k \end{cases} \tag{1–56}
$$

At the derivation of the Cartesian representation it has been used that the base vector $\mathbf{i}_j$ is constant as a function of x^l. In contrast, the curvilinear base vectors depend on the curvilinear coordinates, which accounts for the second term within the parentheses in (1-56). Clearly, $\frac{\partial\mathbf{g}_j}{\partial\theta^k}$ and $\frac{\partial\mathbf{g}^j}{\partial\theta^k}$ are vectors, which may hence be decomposed in the covariant and contravariant vector bases as follows:

$$
\left.\begin{aligned} \frac{\partial\mathbf{g}_j}{\partial\theta^k} &= \left\{ {l \atop j\ \ k} \right\}\mathbf{g}_l \\ \frac{\partial\mathbf{g}^j}{\partial\theta^k} &= -\left\{ {j \atop l\ \ k} \right\}\mathbf{g}^l \end{aligned}\right\} \tag{1–57}
$$

$\left\{ {l \atop j\ \ k} \right\}$ signify the covariant components of $\frac{\partial\mathbf{g}_j}{\partial\theta^k}$, and $-\left\{ {j \atop l\ \ k} \right\}$ is the contravariant components of $\frac{\partial\mathbf{g}^j}{\partial\theta^k}$. $\left\{ {l \atop j\ \ k} \right\}$ is denoted the *Christoffel symbol*. This is related to the co- and contravariant components of the identity tensor as:

$$
\left\{ {l \atop j\ \ k} \right\} = \frac{1}{2}\,g^{lm}\left(\frac{\partial g_{jm}}{\partial\theta^k} + \frac{\partial g_{km}}{\partial\theta^j} - \frac{\partial g_{jk}}{\partial\theta^m}\right) \tag{1–58}
$$

(1-57) and (1-58) have been proved in Box 1.1. From (1-22) and (1-58) follows that the Christoffel symbols fulfill the symmetry condition:

$$
\left\{ {l \atop j\ \ k} \right\} = \left\{ {l \atop k\ \ j} \right\} \tag{1–59}
$$

From (1-8) follows that:

$$
\frac{\partial\mathbf{g}_j}{\partial\theta^k} = \frac{\partial^2\mathbf{x}}{\partial\theta^j\partial\theta^k} = \frac{\partial^2\mathbf{x}}{\partial\theta^k\partial\theta^j} = \frac{\partial\mathbf{g}_k}{\partial\theta^j} \tag{1–60}
$$

Alternatively, the symmetry property (1-59) may be proved by insertion of (1-57) in (1-60).

Box 1.1: Proof of (1-57) and (1-58)

Consider the first relation (1-57) as a definition of the Christoffel symbol, and prove that this implies the second relation (1-57).

From (1-15) and the first relation (1-57) follows:

$$\frac{\partial}{\partial \theta^k}\delta_m^j = \frac{\partial}{\partial \theta^k}\big(\mathbf{g}_m \cdot \mathbf{g}^j\big) \;=\; \frac{\partial \mathbf{g}_m}{\partial \theta^k} \cdot \mathbf{g}^j + \mathbf{g}_m \cdot \frac{\partial \mathbf{g}^j}{\partial \theta^k} \;=\; 0 \quad \Rightarrow$$

$$\frac{\partial \mathbf{g}^j}{\partial \theta^k} \cdot \mathbf{g}_m \;=\; -\begin{Bmatrix} l \\ m \;\; k \end{Bmatrix} \mathbf{g}_l \cdot \mathbf{g}^j \;=\; -\begin{Bmatrix} l \\ m \;\; k \end{Bmatrix} \delta_l^j \;=\; -\begin{Bmatrix} j \\ m \;\; k \end{Bmatrix} \;=\; -\begin{Bmatrix} j \\ l \;\; k \end{Bmatrix} \delta_m^l \;=$$

$$-\begin{Bmatrix} j \\ l \;\; k \end{Bmatrix} \mathbf{g}^l \cdot \mathbf{g}_m \qquad \Rightarrow$$

$$\left(\frac{\partial \mathbf{g}^j}{\partial \theta^k} + \begin{Bmatrix} j \\ l \;\; k \end{Bmatrix} \mathbf{g}^l\right) \cdot \mathbf{g}_m = 0 \tag{1–61}$$

Since (1-61) is valid for any of the N linear independent covariant base vectors $\mathbf{g}_m$, the term within the bracket must be equal to $\mathbf{0}$. This proves the validity of the second relation (1-57).

From (1-22) and the first relation (1-57) follows:

$$\frac{\partial g_{jm}}{\partial \theta^k} \;=\; \frac{\partial(\mathbf{g}_j \cdot \mathbf{g}_m)}{\partial \theta^k} \;=\; \frac{\partial \mathbf{g}_j}{\partial \theta^k} \cdot \mathbf{g}_m + \frac{\partial \mathbf{g}_m}{\partial \theta^k} \cdot \mathbf{g}_j \;=\; \begin{Bmatrix} n \\ j \;\; k \end{Bmatrix} g_{nm} + \begin{Bmatrix} n \\ m \;\; k \end{Bmatrix} g_{nj} \tag{1–62}$$

From (1-62) follows:

$$\frac{\partial g_{jm}}{\partial \theta^k} + \frac{\partial g_{km}}{\partial \theta^j} - \frac{\partial g_{jk}}{\partial \theta^m} \;=$$

$$\begin{Bmatrix} n \\ j \;\; k \end{Bmatrix} g_{nm} + \begin{Bmatrix} n \\ m \;\; k \end{Bmatrix} g_{nj} + \begin{Bmatrix} n \\ k \;\; j \end{Bmatrix} g_{nm} + \begin{Bmatrix} n \\ m \;\; j \end{Bmatrix} g_{nk} - \begin{Bmatrix} n \\ j \;\; m \end{Bmatrix} g_{nk} - \begin{Bmatrix} n \\ k \;\; m \end{Bmatrix} g_{nj} \;=$$

$$2\begin{Bmatrix} n \\ j \;\; k \end{Bmatrix} g_{nm} \tag{1–63}$$

where the symmetry property (1-59) has been used. Next, (1-58) follows from (1-63) upon pre-multiplication on both sides with g^{lm} and use of (1-23).

Insertion of (1-57) in (1-56) provides the following representations of $\boldsymbol{\nabla}\mathbf{v}$:

$$\boldsymbol{\nabla}\mathbf{v} = v^j{}_{;k}\,\mathbf{g}_j\mathbf{g}^k = v_{j;k}\,\mathbf{g}^j\mathbf{g}^k \tag{1–64}$$

where:

$$\left.\begin{aligned} v^j{}_{;k} &= \frac{\partial v^j}{\partial \theta^k} + \begin{Bmatrix} j \\ k \;\; l \end{Bmatrix} v^l \\ v_{j;k} &= \frac{\partial v_j}{\partial \theta^k} - \begin{Bmatrix} l \\ j \;\; k \end{Bmatrix} v_l \end{aligned}\right\} \tag{1–65}$$

Hence, $v^j{}_{;k}$ specifies the mixed co-and contravariant components, and $v_{j;k}$ specifies the contravariant components of $\boldsymbol{\nabla}\mathbf{v}$.

By the use of (1-57) the partial derivative of the vector function $\mathbf{v}(\theta^l)$ may be written as:

$$\frac{\partial \mathbf{v}}{\partial \theta^k} = \begin{cases} \dfrac{\partial (v^j \mathbf{g}_j)}{\partial \theta^k} = \dfrac{\partial v^j}{\partial \theta^k}\mathbf{g}_j + \dfrac{\partial \mathbf{g}_j}{\partial \theta^k} v^j = v^j{}_{;k}\,\mathbf{g}_j \\ \dfrac{\partial (v_j \mathbf{g}^j)}{\partial \theta^k} = \dfrac{\partial v_j}{\partial \theta^k}\mathbf{g}^j + \dfrac{\partial \mathbf{g}^j}{\partial \theta^k} v_j = v_{j;k}\,\mathbf{g}^j \end{cases} \tag{1–66}$$

Hence, alternatively $v^j{}_{;k}$ and $v_{j;k}$ may be interpreted as the covariant components and the contravariant components of the vector $\frac{\partial \mathbf{v}}{\partial \theta^k}$. For this reason $v^j{}_{;k}$ is referred to as the *covariant derivative*, and $v_{j;k}$ as the *contravariate derivative* of the components v^j and v_j, respective. In the applied notation these derivatives will always be indicated by a semicolon.

Further, by the use of (1-57) the following results for the derivatives of the dyads entering the covariant, mixed and contravariant tensor bases become:

$$\left.\begin{aligned} \frac{\partial}{\partial \theta^l}(\mathbf{g}_j\mathbf{g}_k) &= \frac{\partial \mathbf{g}_j}{\partial \theta^l}\mathbf{g}_k + \mathbf{g}_j\frac{\partial \mathbf{g}_k}{\partial \theta^l} = \begin{Bmatrix} m \\ j \;\; l \end{Bmatrix} \mathbf{g}_m\mathbf{g}_k + \begin{Bmatrix} m \\ k \;\; l \end{Bmatrix} \mathbf{g}_j\mathbf{g}_m \\ \frac{\partial}{\partial \theta^l}(\mathbf{g}_j\mathbf{g}^k) &= \frac{\partial \mathbf{g}_j}{\partial \theta^l}\mathbf{g}^k + \mathbf{g}_j\frac{\partial \mathbf{g}^k}{\partial \theta^l} = \begin{Bmatrix} m \\ j \;\; l \end{Bmatrix} \mathbf{g}_m\mathbf{g}^k - \begin{Bmatrix} k \\ m \;\; l \end{Bmatrix} \mathbf{g}_j\mathbf{g}^m \\ \frac{\partial}{\partial \theta^l}(\mathbf{g}^j\mathbf{g}_k) &= \frac{\partial \mathbf{g}^j}{\partial \theta^l}\mathbf{g}_k + \mathbf{g}^j\frac{\partial \mathbf{g}_k}{\partial \theta^l} = -\begin{Bmatrix} j \\ m \;\; l \end{Bmatrix} \mathbf{g}^m\mathbf{g}_k + \begin{Bmatrix} m \\ k \;\; l \end{Bmatrix} \mathbf{g}^j\mathbf{g}_m \\ \frac{\partial}{\partial \theta^l}(\mathbf{g}^j\mathbf{g}^k) &= \frac{\partial \mathbf{g}^j}{\partial \theta^l}\mathbf{g}^k + \mathbf{g}^j\frac{\partial \mathbf{g}^k}{\partial \theta^l} = -\begin{Bmatrix} j \\ m \;\; l \end{Bmatrix} \mathbf{g}^m\mathbf{g}^k - \begin{Bmatrix} k \\ m \;\; l \end{Bmatrix} \mathbf{g}^j\mathbf{g}^m \end{aligned}\right\} \tag{1–67}$$

Finally, the covariant and contravariant components of the differential increment $d\mathbf{v}$ of the vector $\mathbf{v}$ due to the differential incremental vector $d\boldsymbol{\theta} = d\theta^k\,\mathbf{g}_k$ of the curvilinear coordinates becomes:

$$d\mathbf{v} = \boldsymbol{\nabla}\mathbf{v}\cdot d\boldsymbol{\theta} = v^j_{\ ;k}\,\mathbf{g}_j\mathbf{g}^k\cdot d\theta^l\,\mathbf{g}_l = v_{j;k}\,\mathbf{g}^j\mathbf{g}^k\cdot d\theta^l\,\mathbf{g}_l \quad\Rightarrow$$
$$d\mathbf{v} = v^j_{\ ;k}\,d\theta^k\,\mathbf{g}_j = v_{j;k}\,d\theta^k\,\mathbf{g}^j \tag{1–68}$$

The *gradient of a second order tensor function* $\mathbf{T} = \mathbf{T}(\theta^l)$, denoted as $\boldsymbol{\nabla}\mathbf{T}$, is a *third order tensor* with the following representations:

$$\boldsymbol{\nabla}\mathbf{T} = \frac{\partial\mathbf{T}}{\partial\theta^l}\,\mathbf{g}^l = \begin{cases} \dfrac{\partial(T^{jk}\,\mathbf{g}_j\mathbf{g}_k)}{\partial\theta^l}\,\mathbf{g}^l = \left(\dfrac{\partial T^{jk}}{\partial\theta^l}\,\mathbf{g}_j\mathbf{g}_k + T^{jk}\,\dfrac{\partial\mathbf{g}_j}{\partial\theta^l}\mathbf{g}_k + T^{jk}\,\mathbf{g}_j\,\dfrac{\partial\mathbf{g}_k}{\partial\theta^l}\right)\mathbf{g}^l \\ \dfrac{\partial(T^j_{\ k}\,\mathbf{g}_j\mathbf{g}^k)}{\partial\theta^l}\,\mathbf{g}^l = \left(\dfrac{\partial T^j_{\ k}}{\partial\theta^l}\,\mathbf{g}_j\mathbf{g}^k + T^j_{\ k}\,\dfrac{\partial\mathbf{g}_j}{\partial\theta^l}\mathbf{g}^k + T^j_{\ k}\,\mathbf{g}_j\,\dfrac{\partial\mathbf{g}^k}{\partial\theta^l}\right)\mathbf{g}^l \\ \dfrac{\partial(T_j^{\ k}\,\mathbf{g}^j\mathbf{g}_k)}{\partial\theta^l}\,\mathbf{g}^l = \left(\dfrac{\partial T_j^{\ k}}{\partial\theta^l}\,\mathbf{g}^j\mathbf{g}_k + T_j^{\ k}\,\dfrac{\partial\mathbf{g}^j}{\partial\theta^l}\mathbf{g}_k + T_j^{\ k}\,\mathbf{g}^j\,\dfrac{\partial\mathbf{g}_k}{\partial\theta^l}\right)\mathbf{g}^l \\ \dfrac{\partial(T_{jk}\,\mathbf{g}^j\mathbf{g}^k)}{\partial\theta^l}\,\mathbf{g}^l = \left(\dfrac{\partial T_{jk}}{\partial\theta^l}\,\mathbf{g}^j\mathbf{g}^k + T_{jk}\,\dfrac{\partial\mathbf{g}^j}{\partial\theta^l}\mathbf{g}^k + T_{jk}\,\mathbf{g}^j\,\dfrac{\partial\mathbf{g}^k}{\partial\theta^l}\right)\mathbf{g}^l \end{cases} \Rightarrow$$

$$\boldsymbol{\nabla}\mathbf{T} = \begin{cases} \left(\dfrac{\partial T^{jk}}{\partial\theta^l} + T^{mk}\begin{Bmatrix} j \\ m\ \ l\end{Bmatrix} + T^{jm}\begin{Bmatrix} k \\ m\ \ l\end{Bmatrix}\right)\mathbf{g}_j\mathbf{g}_k\mathbf{g}^l = T^{jk}_{\ \ ;l}\,\mathbf{g}_j\mathbf{g}_k\mathbf{g}^l \\ \left(\dfrac{\partial T^j_{\ k}}{\partial\theta^l} + T^m_{\ k}\begin{Bmatrix} j \\ m\ \ l\end{Bmatrix} - T^j_{\ m}\begin{Bmatrix} m \\ k\ \ l\end{Bmatrix}\right)\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l = T^j_{\ k;l}\,\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l \\ \left(\dfrac{\partial T_j^{\ k}}{\partial\theta^l} - T_m^{\ k}\begin{Bmatrix} m \\ j\ \ l\end{Bmatrix} + T_j^{\ m}\begin{Bmatrix} k \\ m\ \ l\end{Bmatrix}\right)\mathbf{g}^j\mathbf{g}_k\mathbf{g}^l = T_j^{\ k}{}_{;l}\,\mathbf{g}^j\mathbf{g}_k\mathbf{g}^l \\ \left(\dfrac{\partial T_{jk}}{\partial\theta^l} - T_{km}\begin{Bmatrix} m \\ j\ \ l\end{Bmatrix} - T_{jm}\begin{Bmatrix} m \\ k\ \ l\end{Bmatrix}\right)\mathbf{g}^j\mathbf{g}^k\mathbf{g}^l = T_{jk;l}\,\mathbf{g}^j\mathbf{g}^k\mathbf{g}^l \end{cases} \tag{1–69}$$

where:

$$\left.\begin{aligned} T^{jk}_{\ \ ;l} &= \frac{\partial T^{jk}}{\partial\theta^l} + T^{mk}\begin{Bmatrix} j \\ m\ \ l\end{Bmatrix} + T^{jm}\begin{Bmatrix} k \\ m\ \ l\end{Bmatrix} \\ T^j_{\ k;l} &= \frac{\partial T^j_{\ k}}{\partial\theta^l} + T^m_{\ k}\begin{Bmatrix} j \\ m\ \ l\end{Bmatrix} - T^j_{\ m}\begin{Bmatrix} m \\ k\ \ l\end{Bmatrix} \\ T_j^{\ k}{}_{;l} &= \frac{\partial T_j^{\ k}}{\partial\theta^l} - T_m^{\ k}\begin{Bmatrix} m \\ j\ \ l\end{Bmatrix} + T_j^{\ m}\begin{Bmatrix} k \\ m\ \ l\end{Bmatrix} \\ T_{jk;l} &= \frac{\partial T_{jk}}{\partial\theta^l} - T_{km}\begin{Bmatrix} m \\ j\ \ l\end{Bmatrix} - T_{jm}\begin{Bmatrix} m \\ k\ \ l\end{Bmatrix} \end{aligned}\right\} \tag{1–70}$$

(1-57) has been used in the last statements of (1-69).

Then, the partial derivative of the tensor function $\mathbf{T}(\theta^m)$ may be described as any of the following second order tensor representations, cf. (1-66):

$$\frac{\partial \mathbf{T}}{\partial \theta^l} = T^{jk}{}_{;l}\, \mathbf{g}_j \mathbf{g}_k = T^{j}{}_{k;l}\, \mathbf{g}_j \mathbf{g}^k = T_{jk;l}\, \mathbf{g}^j \mathbf{g}^k \tag{1–71}$$

Partial differentiation of both sides of (1-44) provides:

$$\frac{\partial \mathbf{v}}{\partial \theta^k} = \frac{\partial \mathbf{g}}{\partial \theta^k} \cdot \mathbf{v} + \mathbf{g} \cdot \frac{\partial \mathbf{v}}{\partial \theta^k} = \frac{\partial \mathbf{g}}{\partial \theta^k} \cdot \mathbf{v} + \frac{\partial \mathbf{v}}{\partial \theta^k} \qquad \Rightarrow$$

$$\frac{\partial \mathbf{g}}{\partial \theta^k} \cdot \mathbf{v} = 0 \tag{1–72}$$

Since $\mathbf{v}$ is arbitrary, (1-72) implies that:

$$\frac{\partial \mathbf{g}}{\partial \theta^k} = \mathbf{0} \tag{1–73}$$

In turn this means that $\boldsymbol{\nabla} \mathbf{g} = \frac{\partial \mathbf{g}}{\partial \theta^l}\, \mathbf{g}^l = \mathbf{0}$. Then, from (1-69) follows that $g^{jk}{}_{;l} = g_{jk;l} = 0$, so the components of the identity tensor vanish under covariant and contravariant differentiation.

1.4 Riemann-Christoffel tensor

The gradient $\boldsymbol{\nabla}\mathbf{v}$ of a vector $\mathbf{v}$ with components in Cartesian and curvilinear tensor bases have been indicated by (1-56). The gradient of this second order tensor is given as, cf. (1-69):

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{v}) = \begin{cases} \dfrac{\partial}{\partial x^l}\left(\dfrac{\partial \mathbf{v}}{\partial x^k}\,\mathbf{i}_k\right)\mathbf{i}_l = \dfrac{\partial}{\partial x^l}\left(\dfrac{\partial (\bar{v}^j\,\mathbf{i}_j)}{\partial x^k}\,\mathbf{i}_k\right)\mathbf{i}_l = \dfrac{\partial^2 \bar{v}^j}{\partial x^k \partial x^l}\,\mathbf{i}_j\mathbf{i}_k\mathbf{i}_l \\[2ex] \dfrac{\partial}{\partial \theta^l}\left(\dfrac{\partial \mathbf{v}}{\partial \theta^k}\,\mathbf{g}^k\right)\mathbf{g}^l = \dfrac{\partial}{\partial \theta^l}\left(\left(\dfrac{\partial v^j}{\partial \theta^k} + \begin{Bmatrix} j \\ k \;\; m \end{Bmatrix} v^m\right)\mathbf{g}_j\mathbf{g}^k\right)\mathbf{g}^l = v^j{}_{;kl}\,\mathbf{g}_j\mathbf{g}^k\mathbf{g}^l \\[2ex] \dfrac{\partial}{\partial \theta^l}\left(\dfrac{\partial \mathbf{v}}{\partial \theta^k}\,\mathbf{g}^k\right)\mathbf{g}^l = \dfrac{\partial}{\partial \theta^l}\left(\left(\dfrac{\partial v_j}{\partial \theta^k} - \begin{Bmatrix} m \\ j \;\; k \end{Bmatrix} v_m\right)\mathbf{g}^j\mathbf{g}^k\right)\mathbf{g}^l = v_{j;kl}\,\mathbf{g}^j\mathbf{g}^k\mathbf{g}^l \end{cases} \tag{1–74}$$

where the following tensor components have been introduced:

$$v^j{}_{;kl} = (v^j{}_{;k})_{;l} = \frac{\partial^2 v^j}{\partial \theta^k \partial \theta^l} + \begin{Bmatrix} j \\ k \;\; m \end{Bmatrix} \frac{\partial v^m}{\partial \theta^l} + \begin{Bmatrix} j \\ l \;\; m \end{Bmatrix} \frac{\partial v^m}{\partial \theta^k} - \begin{Bmatrix} m \\ k \;\; l \end{Bmatrix} \frac{\partial v^j}{\partial \theta^m} -$$
$$\begin{Bmatrix} n \\ k \;\; l \end{Bmatrix} \begin{Bmatrix} j \\ n \;\; m \end{Bmatrix} v^m + \frac{\partial}{\partial \theta^l} \begin{Bmatrix} j \\ k \;\; m \end{Bmatrix} v^m + \begin{Bmatrix} j \\ l \;\; n \end{Bmatrix} \begin{Bmatrix} n \\ k \;\; m \end{Bmatrix} v^m \tag{1–75}$$

$$v_{j;kl} = (v_{j;k})_{;l} = \frac{\partial^2 v_j}{\partial \theta^k \partial \theta^l} - \begin{Bmatrix} m \\ j \;\; k \end{Bmatrix} \frac{\partial v_m}{\partial \theta^l} - \begin{Bmatrix} m \\ j \;\; l \end{Bmatrix} \frac{\partial v_m}{\partial \theta^k} - \begin{Bmatrix} m \\ k \;\; l \end{Bmatrix} \frac{\partial v_j}{\partial \theta^m} +$$
$$\begin{Bmatrix} n \\ k \;\; l \end{Bmatrix} \begin{Bmatrix} m \\ n \;\; j \end{Bmatrix} v_m - \frac{\partial}{\partial \theta^l} \begin{Bmatrix} m \\ j \;\; k \end{Bmatrix} v_m + \begin{Bmatrix} n \\ j \;\; l \end{Bmatrix} \begin{Bmatrix} m \\ n \;\; k \end{Bmatrix} v_m \tag{1–76}$$

From (1-75) and (1-76) follow that the indices k and l can be interchanged in the first five terms on the right hand sides without changing the value of this part of the expressions, whereas this is not the case for the last two terms. This implies that the sequence in which the covariant differentiations is performed is significant, i.e. in general $v_{j;kl} \neq v_{j;lk}$. In order to investigate this further consider the quantity:

$$v_{j;kl} - v_{j;kl} = \left(\begin{Bmatrix} n \\ j \;\; l \end{Bmatrix} \begin{Bmatrix} m \\ n \;\; k \end{Bmatrix} - \begin{Bmatrix} n \\ j \;\; k \end{Bmatrix} \begin{Bmatrix} m \\ n \;\; l \end{Bmatrix} + \frac{\partial}{\partial \theta^k} \begin{Bmatrix} m \\ j \;\; l \end{Bmatrix} - \frac{\partial}{\partial \theta^l} \begin{Bmatrix} m \\ j \;\; k \end{Bmatrix} \right) v_m = R^m{}_{jkl}\, v_m \tag{1–77}$$

where:

$$R^m{}_{jkl} = \begin{Bmatrix} n \\ j \;\; l \end{Bmatrix} \begin{Bmatrix} m \\ n \;\; k \end{Bmatrix} - \begin{Bmatrix} n \\ j \;\; k \end{Bmatrix} \begin{Bmatrix} m \\ n \;\; l \end{Bmatrix} + \frac{\partial}{\partial \theta^k} \begin{Bmatrix} m \\ j \;\; l \end{Bmatrix} - \frac{\partial}{\partial \theta^l} \begin{Bmatrix} m \\ j \;\; k \end{Bmatrix} \tag{1–78}$$

$R^m{}_{jkl}$ signifies the mixed components of the so-called *Riemann-Christoffel tensor* $\mathbf{R}$, i.e. $\mathbf{R} = R^m{}_{jkl}\, \mathbf{g}_m \mathbf{g}^j \mathbf{g}^k \mathbf{g}^l$. The components of $\mathbf{R}$, and hence the right hand side of (1-77), are not vanishing due to the curvature of the Riemann space.

Obviously,

$$R^m{}_{jkl} = -R^m{}_{jlk} \tag{1–79}$$

Further, the following socalled *Bianchi's first identity* applies:

$$R^m{}_{jkl} + R^m{}_{klj} + R^m{}_{ljk} = 0 \tag{1–80}$$

(1-80) follows by insertion of (1-78) and use of the symmetry property (1-59) of the Christoffel symbol.

In an Euclidean N-dimensional space, i.e. a space spanned by a constant Cartesian vector basis, the Cartesian components of $\mathbf{R}$ follow from (1-74):

$$\bar{R}^{jklm}\, \bar{v}_m = \frac{\partial^2 \bar{v}^j}{\partial x^k \partial x^l} - \frac{\partial^2 \bar{v}^j}{\partial x^l \partial x^k} = 0 \qquad \Rightarrow \qquad \bar{R}^{jklm} = 0 \tag{1–81}$$

Hence, it can be concluded that $\mathbf{R} = \mathbf{0}$ in an Euclidean space. In turn this means that the curvilinear components (1-78) must also vanish in this space. A space, where everywhere $\mathbf{R} = \mathbf{0}$ is called *flat*. Reversely, a non-vanishing curvature tensor indicates a curved space. In a flat space $R^m{}_{jkl} = 0$, with the implication that $v_{j;kl} = v_{j;lk}$. The three-dimensional Euclidian space is flat, and any plane in this space forms a flat two-dimensional subspace. In contrast, a curved surface in the Euclidian space is not a flat subspace. An example of a curved four-dimensional space is the time-space manifold used at the formulation of the general theory of relativity, where the indices correspondingly range over $j = 1, 2, 3, 4$.

Because of the relations (1-79), (1-80) only $\frac{1}{12}N^2(N^2-1)$ of the tensor components $R^m{}_{jkl}$ are independent and non-trivial, (Spain, 1965). Hence, for a two-dimensional Riemann space merely one independent component exists, which can be chosen as $R^1{}_{212}$. In the three-dimensional case six independent and non-trivial components exist, which may be chosen as $R^1{}_{212}$, $R^1{}_{213}$, $R^1{}_{223}$, $R^1{}_{313}$, $R^1{}_{323}$ and $R^2{}_{323}$.

Example 1.1: Covariant base vectors, identity tensor, Christoffel symbols and Riemann-Christoffel tensor in spherical coordinates

By the use of (1-10) the first equation (1-57) becomes:

$$
\frac{\partial(c_j^m\,\mathbf{i}_m)}{\partial\theta^k} = \frac{\partial c_j^m}{\partial\theta^k}\,\mathbf{i}_m = \begin{Bmatrix} l \\ j \;\; k \end{Bmatrix} c_l^m\,\mathbf{i}_m \qquad \Rightarrow
$$

$$
\frac{\partial c_j^m}{\partial\theta^k} = \begin{Bmatrix} l \\ j \;\; k \end{Bmatrix} c_l^m \tag{1–82}
$$

The Cartesian components c_j^m of the covariant base vector $\mathbf{g}_j$ is stored in the column matrix $\underline{g}_j = [c_j^m]$. Then, (1-82) may be written in the following matrix form:

$$
\frac{\partial\underline{g}_j}{\partial\theta^k} = \begin{Bmatrix} l \\ j \;\; k \end{Bmatrix} \underline{g}_l = \begin{Bmatrix} 1 \\ j \;\; k \end{Bmatrix} \underline{g}_1 + \begin{Bmatrix} 2 \\ j \;\; k \end{Bmatrix} \underline{g}_2 + \begin{Bmatrix} 3 \\ j \;\; k \end{Bmatrix} \underline{g}_3 \tag{1–83}
$$

The spherical coordinate system defined by (1-1) is considered. In this case the column matrices attain the form, cf. (1-8):

$$
\underline{g}_1 = \begin{bmatrix} \theta^3\cos\theta^1\cos\theta^2 \\ \theta^3\cos\theta^1\sin\theta^2 \\ -\theta^3\sin\theta^1 \end{bmatrix} \quad , \quad \underline{g}_2 = \begin{bmatrix} -\theta^3\sin\theta^1\sin\theta^2 \\ \theta^3\sin\theta^1\cos\theta^2 \\ 0 \end{bmatrix} \quad , \quad \underline{g}_3 = \begin{bmatrix} \sin\theta^1\cos\theta^2 \\ \sin\theta^1\sin\theta^2 \\ \cos\theta^1 \end{bmatrix} \tag{1–84}
$$

The covariant components of the identity tensor is given by (1-22) as $g_{jk} = \mathbf{g}_j\cdot\mathbf{g}_k = \underline{g}_j^T\underline{g}_k$, where the last statement is obtained by evaluating the scalar product in Cartesian coordinates. The covariant and contravariant components of the identity tensor are conveniently stored in matrices. Using (1-84) these becomes:

$$
[\,g_{jk}\,] = \begin{bmatrix} (\theta^3)^2 & 0 & 0 \\ 0 & (\theta^3)^2\sin^2\theta^1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad [\,g^{jk}\,] = \begin{bmatrix} \dfrac{1}{(\theta^3)^2} & 0 & 0 \\ 0 & \dfrac{1}{(\theta^3)^2\sin^2\theta^1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1–85}
$$

The result for the contravariant components follows from (1-23). Next, (1-83) and (1-84) will be used to determine the Christoffel. The following results are obtained:

$$\left.\begin{array}{lll}
\dfrac{\partial \underline{g}_1}{\partial \theta^1} = \begin{bmatrix} -\theta^3 \sin\theta^1 \cos\theta^2 \\ -\theta^3 \sin\theta^1 \sin\theta^2 \\ -\theta^3 \cos\theta^1 \end{bmatrix} = 0 \cdot \underline{g}_1 + 0 \cdot \underline{g}_2 - \theta^3 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 1\ 1} \right\} = 0 \\ \left\{ {2 \atop 1\ 1} \right\} = 0 \\ \left\{ {3 \atop 1\ 1} \right\} = -\theta^3 \end{cases} \\
\dfrac{\partial \underline{g}_1}{\partial \theta^2} = \begin{bmatrix} -\theta^3 \cos\theta^1 \sin\theta^2 \\ \theta^3 \cos\theta^1 \cos\theta^2 \\ 0 \end{bmatrix} = 0 \cdot \underline{g}_1 + \dfrac{1}{\tan\theta^1} \cdot \underline{g}_2 + 0 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 1\ 2} \right\} = 0 \\ \left\{ {2 \atop 1\ 2} \right\} = \dfrac{1}{\tan\theta^1} \\ \left\{ {3 \atop 1\ 2} \right\} = 0 \end{cases} \\
\dfrac{\partial \underline{g}_1}{\partial \theta^3} = \begin{bmatrix} \cos\theta^1 \cos\theta^2 \\ \cos\theta^1 \sin\theta^2 \\ -\sin\theta^1 \end{bmatrix} = \dfrac{1}{\theta^3} \cdot \underline{g}_1 + 0 \cdot \underline{g}_2 + 0 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 1\ 3} \right\} = \dfrac{1}{\theta^3} \\ \left\{ {2 \atop 1\ 3} \right\} = 0 \\ \left\{ {3 \atop 1\ 3} \right\} = 0 \end{cases} \\
\dfrac{\partial \underline{g}_2}{\partial \theta^1} = \begin{bmatrix} -\theta^3 \cos\theta^1 \sin\theta^2 \\ \theta^3 \cos\theta^1 \cos\theta^2 \\ 0 \end{bmatrix} = 0 \cdot \underline{g}_1 + \dfrac{1}{\tan\theta^1} \cdot \underline{g}_2 + 0 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 2\ 1} \right\} = 0 \\ \left\{ {2 \atop 2\ 1} \right\} = \dfrac{1}{\tan\theta^1} \\ \left\{ {3 \atop 2\ 1} \right\} = 0 \end{cases} \\
\dfrac{\partial \underline{g}_2}{\partial \theta^2} = \begin{bmatrix} -\theta^3 \sin\theta^1 \cos\theta^2 \\ -\theta^3 \sin\theta^1 \sin\theta^2 \\ 0 \end{bmatrix} = -\dfrac{1}{2}\sin(2\theta^1) \cdot \underline{g}_1 + 0 \cdot \underline{g}_2 - \theta^3 \sin^2\theta^1 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 2\ 2} \right\} = -\dfrac{1}{2}\sin(2\theta^1) \\ \left\{ {2 \atop 2\ 2} \right\} = 0 \\ \left\{ {3 \atop 2\ 2} \right\} = -\theta^3 \sin^2\theta^1 \end{cases} \\
\dfrac{\partial \underline{g}_2}{\partial \theta^3} = \begin{bmatrix} -\sin\theta^1 \sin\theta^2 \\ \sin\theta^1 \cos\theta^2 \\ 0 \end{bmatrix} = 0 \cdot \underline{g}_1 + \dfrac{1}{\theta^3} \cdot \underline{g}_2 + 0 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 2\ 3} \right\} = 0 \\ \left\{ {2 \atop 2\ 3} \right\} = \dfrac{1}{\theta^3} \\ \left\{ {3 \atop 2\ 3} \right\} = 0 \end{cases} \\
\dfrac{\partial \underline{g}_3}{\partial \theta^1} = \begin{bmatrix} \cos\theta^1 \cos\theta^2 \\ \cos\theta^1 \sin\theta^2 \\ -\sin\theta^1 \end{bmatrix} = \dfrac{1}{\theta^3} \cdot \underline{g}_1 + 0 \cdot \underline{g}_2 + 0 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 3\ 1} \right\} = \dfrac{1}{\theta^3} \\ \left\{ {2 \atop 3\ 1} \right\} = 0 \\ \left\{ {3 \atop 3\ 1} \right\} = 0 \end{cases} \\
\dfrac{\partial \underline{g}_3}{\partial \theta^2} = \begin{bmatrix} -\sin\theta^1 \sin\theta^2 \\ \sin\theta^1 \cos\theta^2 \\ 0 \end{bmatrix} = 0 \cdot \underline{g}_1 + \dfrac{1}{\theta^3} \cdot \underline{g}_2 + 0 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 3\ 2} \right\} = 0 \\ \left\{ {2 \atop 3\ 2} \right\} = \dfrac{1}{\theta^3} \\ \left\{ {3 \atop 3\ 2} \right\} = 0 \end{cases} \\
\dfrac{\partial \underline{g}_3}{\partial \theta^3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \underline{g}_1 + 0 \cdot \underline{g}_2 + 0 \cdot \underline{g}_3 & \Rightarrow & \begin{cases} \left\{ {1 \atop 3\ 3} \right\} = 0 \\ \left\{ {2 \atop 3\ 3} \right\} = 0 \\ \left\{ {3 \atop 3\ 3} \right\} = 0 \end{cases}
\end{array}\right\} \tag{1–86}$$

The non-trivial components of the Riemann-Christoffel tensor follow from (1-77) and (1-86):

$$
\left.
\begin{aligned}
&R^1{}_{212} = \begin{Bmatrix} n \\ 2\;\;2 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;1 \end{Bmatrix} - \begin{Bmatrix} n \\ 2\;\;1 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;2 \end{Bmatrix} + \frac{\partial}{\partial \theta^1} \begin{Bmatrix} 1 \\ 2\;\;2 \end{Bmatrix} - \frac{\partial}{\partial \theta^2} \begin{Bmatrix} 1 \\ 2\;\;1 \end{Bmatrix} = \\
&-\frac{1}{2}\sin(2\theta^1)\cdot 0 + 0\cdot 0 - \theta^3 \sin^2\theta^1 \cdot \frac{1}{\theta^3} + 0\cdot 0 + \frac{1}{\tan\theta^1}\cdot\frac{1}{2}\sin(2\theta^1) - 0\cdot 0 - \cos(2\theta^1) - 0 = 0 \\
&R^1{}_{213} = \begin{Bmatrix} n \\ 2\;\;3 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;1 \end{Bmatrix} - \begin{Bmatrix} n \\ 2\;\;1 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;3 \end{Bmatrix} + \frac{\partial}{\partial \theta^1} \begin{Bmatrix} 1 \\ 2\;\;3 \end{Bmatrix} - \frac{\partial}{\partial \theta^3} \begin{Bmatrix} 1 \\ 2\;\;1 \end{Bmatrix} = \\
&0\cdot 0 + \frac{1}{\theta^3}\cdot 0 + 0\cdot\frac{1}{\theta^3} - 0\cdot\frac{1}{\theta^3} - \frac{1}{\tan\theta^1}\cdot 0 - 0\cdot 0 + 0 - 0 = 0 \\
&R^1{}_{223} = \begin{Bmatrix} n \\ 2\;\;3 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;2 \end{Bmatrix} - \begin{Bmatrix} n \\ 2\;\;2 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;3 \end{Bmatrix} + \frac{\partial}{\partial \theta^2} \begin{Bmatrix} 1 \\ 2\;\;3 \end{Bmatrix} - \frac{\partial}{\partial \theta^3} \begin{Bmatrix} 1 \\ 2\;\;2 \end{Bmatrix} = \\
&0\cdot 0 - \frac{1}{\theta^3}\cdot\frac{1}{2}\sin(2\theta^1) + 0\cdot 0 + \frac{1}{2}\sin(2\theta^1)\cdot\frac{1}{\theta^3} - 0\cdot 0 + \theta^3\sin^2\theta^1\cdot 0 + 0 - 0 = 0 \\
&R^1{}_{313} = \begin{Bmatrix} n \\ 3\;\;3 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;1 \end{Bmatrix} - \begin{Bmatrix} n \\ 3\;\;1 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;3 \end{Bmatrix} + \frac{\partial}{\partial \theta^1} \begin{Bmatrix} 1 \\ 3\;\;3 \end{Bmatrix} - \frac{\partial}{\partial \theta^3} \begin{Bmatrix} 1 \\ 3\;\;1 \end{Bmatrix} = \\
&0\cdot 0 + 0\cdot 0 + 0\cdot\frac{1}{\theta^3} - \frac{1}{\theta^3}\cdot\frac{1}{\theta^3} - 0\cdot 0 - 0\cdot 0 + \frac{1}{(\theta^3)^2} = 0 \\
&R^1{}_{323} = \begin{Bmatrix} n \\ 3\;\;3 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;2 \end{Bmatrix} - \begin{Bmatrix} n \\ 3\;\;2 \end{Bmatrix} \begin{Bmatrix} 1 \\ n\;\;3 \end{Bmatrix} + \frac{\partial}{\partial \theta^2} \begin{Bmatrix} 1 \\ 3\;\;3 \end{Bmatrix} - \frac{\partial}{\partial \theta^3} \begin{Bmatrix} 1 \\ 3\;\;2 \end{Bmatrix} = \\
&0\cdot 0 - 0\cdot\frac{1}{2}\sin(2\theta^1) + 0\cdot 0 - 0\cdot\frac{1}{\theta^3} - 0\cdot 0 - 0\cdot 0 + 0 - 0 = 0 \\
&R^2{}_{323} = \begin{Bmatrix} n \\ 3\;\;3 \end{Bmatrix} \begin{Bmatrix} 2 \\ n\;\;2 \end{Bmatrix} - \begin{Bmatrix} n \\ 3\;\;2 \end{Bmatrix} \begin{Bmatrix} 2 \\ n\;\;3 \end{Bmatrix} + \frac{\partial}{\partial \theta^2} \begin{Bmatrix} 2 \\ 3\;\;3 \end{Bmatrix} - \frac{\partial}{\partial \theta^3} \begin{Bmatrix} 2 \\ 3\;\;2 \end{Bmatrix} = \\
&0\cdot\frac{1}{\tan\theta^1} + 0\cdot 0 + 0\cdot\frac{1}{\theta^3} - 0\cdot 0 - \frac{1}{\theta^3}\cdot\frac{1}{\theta^3} - 0\cdot 0 + 0 + \frac{1}{(\theta^3)^2} = 0
\end{aligned}
\right\}
\tag{1–87}
$$

As expected all components of the Riemann-Christoffel tensor vanish as a consequence of the flatness of the three-dimensional Euclidean space.

1.5 Geodesics

In Euclidean 3-dimensional space the shortest distance between two points A and B is a straight line. On a surface embedded in the Euclidian 3-dimensional space, where both principal curvatures everywhere are either positive or negative, the curve with the shortest distance between the points is indicated by an inflexible string stretched between the points. Especially, on a sphere the curve makes up a part of a great circle. In this section this principle is carried over to a general Riemann space in terms of a socalled *geodesics*, which is defined as the curve with the minimum length connecting two points in the Riemann space with the length measured by the fundamental tensor of the space. A geodesics joining the point A and B must fulfill the following variational principle:

$$\delta \int_A^B ds \;=\; 0 \tag{1–88}$$

where ds is a differential length element along an arbitrary curve connecting the points A and B given as:

$$ds^2 \;=\; d\mathbf{x}\cdot d\mathbf{x} \;=\; dx^j dx^k\, \mathbf{g}_j \cdot \mathbf{g}_k \;=\; g_{jk}\, dx^j dx^k \tag{1–89}$$

dx^j indicates the covariant components of the differential increment d$\mathbf{x}$ of the position vector $\mathbf{x}$ tangential to the arc length increment. Further, (1-22) has been used in the last statement.

The geodesics turns out to be given by the following non-linear differential equation:

$$\frac{d^2x^j}{ds^2} + \begin{Bmatrix} j \\ k \;\; l \end{Bmatrix} \frac{dx^k}{ds}\frac{dx^l}{ds} \;=\; 0 \tag{1–90}$$

where x^j indicates the coordinates in a given referential curvilinear coordinate system in the N-dimensional Riemann space of a running point along the geodesic. Eq. (1-90) is solved with the initial value x^j_A and the unit tangential vector $t^j_A = \frac{dx^j_A}{ds}$ specified at a given point point A on the geodesics. A proof of (1-90) is given in Box 1.2.

Consider a point A on a differential surface embedded in the three-dimensional Euclidian space. At an arbitrary point A on the surface two linear independent tangential vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ may be defined, which span the tangent plane at A. The tangent plane is a two-dimensional flat space Euclidian subspace, in contrast to the underlying differential surface. As a consequence the Riemann tensor vanish in the tangential subspace. In the following this observation is generalized from a two-dimensional to an arbitrary N-dimensional curved Riemann space.

Consider the N linear independent unit tangential vectors $\mathbf{t}_j$, $j \;=\; 1,\ldots,N$, indicating the direction of the geodesics drawn out from an arbitrary point Ain the Riemann space. The vectors may be organized to form a local arc length vector basis known as the *Riemann vector basis*, which span a tangential manifold to the Riemann space at A. The related fundamental tensor $\mathbf{g}'$ has the contravariant components, cf. (1-22):

$$g'_{jk} \;=\; \mathbf{t}_j \cdot \mathbf{t}_k \tag{1–91}$$

Let x'^j denote the covariant components in the local Riemann vector basis of a position vector along a geodesic curve. The differential equation of the geodesic in Riemann coordinates reads:

$$\frac{d^2x'^j}{ds^2} + \begin{Bmatrix} j \\ k \;\; l \end{Bmatrix}' \frac{dx'^k}{ds}\frac{dx'^l}{ds} \;=\; 0 \tag{1–92}$$

where $\left\{ {j \atop k\;\; l} \right\}'$ indicates the Christoffel symbol evaluated by the co- and contravariant components of the fundamental tensor $\mathbf{g}'$.

Consider a geodesic curve at the point A defined by the covariant components t^j in the Riemann vector basis with origin at A, and let P be a neighbouring point placed on the geodesics defined by the unit tangent t^j a small arc length s from A. Then, the covariant coordinates of the point P is approximately given as:

$$x'^j \simeq s\,t^j \tag{1–93}$$

(1-93) holds asymptotically as $s \to 0$. Insertion of (1-93) into (1-92) provides in the limit:

$$\begin{Bmatrix} j \\ k \;\; l \end{Bmatrix}' t^j\, t^k = 0 \tag{1–94}$$

The unit tangential vector with the curvilinear components t^j in (1-94) has been arbitrarily selected. Hence, this equation must be fulfilled for all N unit tangential vectors related to the geodesics drawn out of point A. This can only hold, if the Christoffel symbol vanishes at the origin of the Riemann coordinate system at the running point A, i.e.:

$$\begin{Bmatrix} j \\ k \;\; l \end{Bmatrix}' = 0 \tag{1–95}$$

As consequence of (1-95), *the covariant derivatives with respect to x'^j becomes equal to partial derivatives*, cf. (1-65).

Further, from (1-62) follows that:

$$\frac{\partial g'_{jm}}{\partial x'^k} = 0 \tag{1–96}$$

Finally, in a Riemannian coordinate system we have, cf. (1-78):

$$R'^m{}_{jkl;n} = \frac{\partial^2}{\partial x'^n\, \partial x'^k} \begin{Bmatrix} m \\ j \;\; l \end{Bmatrix}' - \frac{\partial^2}{\partial x'^n\, \partial x'^l} \begin{Bmatrix} m \\ j \;\; k \end{Bmatrix}' \tag{1–97}$$

Bianchi's second identity reads:

$$R^m{}_{jkl;n} + R^m{}_{jln;k} + R^m{}_{jnk;l} = 0 \tag{1–98}$$

(1-98) is proved in a Riemannian coordinate system by the use of (1-97). Then, being valid in one coordinate system it is also valid in any curvilinear coordinate system.

Box 1.2: Proof of (1-90)

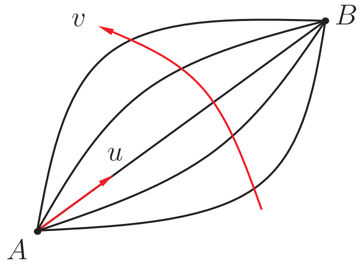

Fig. 1-3: Family of curves connecting points A and B.

Consider a family of curves connecting two point A and B specified by the parametrization:

$$x^j = x^j(u, v) \tag{1–99}$$

The parameter v characterizes a certain curve in the family, and $u \in [a_A, u_B]$ is a parameter defining a certain point on the curve specified by v. Especially, u may be chosen as the length parameter s along the curve. Then the length L of the curve defined by v is given as:

$$L = \int_{u_A}^{u_B} w^{\frac{1}{2}}\, du \tag{1–100}$$

where w is given as, cf. (1-89):

$$w = g_{jk}\, t^j t^k \tag{1–101}$$

and t^j signifies the quantity:

$$t^j = \frac{dx^j}{du} \tag{1–102}$$

Hence, w is a function of the independent variables x^j and t^j.

δL denotes the variation of the length of the curve defined by the parameters (u, v) due to a variation δv of v for fixed u. Then (1-88) attains the form:

$$\delta L = \delta \int_{u_A}^{u_B} w^{\frac{1}{2}}\, du = \int_{u_A}^{u_B} \frac{\partial w^{\frac{1}{2}}}{\partial v}\, \delta v\, du = \int_{u_A}^{u_B} \left(\frac{\partial w^{\frac{1}{2}}}{\partial x^j} \frac{\partial x^j}{\partial v} + \frac{\partial w^{\frac{1}{2}}}{\partial t^j} \frac{\partial t^j}{\partial v} \right) \delta v\, du = 0 \tag{1–103}$$

From (1-102) follows:

$$\frac{\partial t^j}{\partial v} = \frac{d}{du} \frac{\partial x^j}{\partial v} \tag{1–104}$$

Insertion of (1-104) in (1-103) and followed by integration by part provides:

$$\delta L = \left[\frac{\partial w^{\frac{1}{2}}}{\partial t^j} \frac{\partial x^j}{\partial v} \right]_{u_A}^{u_B} + \int_{u_A}^{u_B} \left(\frac{\partial w^{\frac{1}{2}}}{\partial x^j} - \frac{d}{du} \frac{\partial w^{\frac{1}{2}}}{\partial t^j} \right) \delta x^j \, du = 0 \tag{1–105}$$

where $\delta x^j = \frac{\partial x^j}{\partial v} \delta v$ has been introduced.

$x^j(u_A)$ and $x^j(u_B)$ are common for all curve, and hence independent of v. Then, $\frac{\partial x^j(u_A)}{\partial v} = \frac{\partial x^j(u_B)}{\partial v} = 0$, so the boundary terms in (1-105) vanish.

In the integrand δx^j can be varied independently for any $u \in]u_A, u_B[$. This leads to the following *Euler condition* necessary for stationarity:

$$\frac{\partial w^{\frac{1}{2}}}{\partial x^j} - \frac{d}{du} \frac{\partial w^{\frac{1}{2}}}{\partial t^j} = 0 \tag{1–106}$$

(1-106) may be rewritten on the form:

$$\frac{d}{du} \frac{\partial w}{\partial t^j} - \frac{\partial w}{\partial x^j} = \frac{1}{2w} \frac{dw}{du} \frac{\partial w}{\partial t^j} \tag{1–107}$$

Especially, let u be chosen as the arc-length s along the geodesic, so that:

$$u = s \quad , \quad t^j = \frac{dx^j}{ds} \quad , \quad w = g_{jk} \, t^j t^k \equiv 1 \Rightarrow \frac{dw}{du} = 0 \tag{1–108}$$

Now, t^j indicates the covariant components of the unit tangential vector along the geodesic. Insertion of (1-108) in (1-107) provides:

$$\begin{aligned}
&\frac{d}{ds} \frac{\partial w}{\partial t^j} - \frac{\partial w}{\partial x^j} = 0 && \Rightarrow \\
&2 \frac{d}{ds} \left(g_{jk} \, t^k \right) - \frac{\partial g_{kl}}{\partial x^j} t^k t^l = && \Rightarrow \\
&g_{jk} \frac{d^2 x^k}{ds^2} + \frac{\partial g_{jk}}{\partial x^l} \frac{dx^l}{ds} \frac{dx^k}{ds} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^j} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0
\end{aligned} \tag{1–109}$$

where it has been used that $g_{jk} = g_{jk}(x^l)$, so $\frac{dg_{jk}}{ds} = \frac{\partial g_{jk}}{\partial x^l} \frac{dx^l}{ds}$. Further, due to the symmetry property (1-22) of the components of the metric tensor it follows by interchanging the name of the dummy indices l and k that $\frac{\partial g_{jk}}{\partial x^l} \frac{dx^l}{ds} \frac{dx^k}{ds} = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^k} \right) \frac{dx^k}{ds} \frac{dx^l}{ds}$. Then, (1-109) may be written:

$$g_{jk} \frac{d^2 x^k}{ds^2} + \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^j} \right) \frac{dx^k}{ds} \frac{dx^l}{ds} = 0 \tag{1–110}$$

Finally, Eq. (1-90) follows by pre-multiplication and contraction on both sides of (1-110) with g^{mj}, and use of (1-43) and (1-58).

1.6 Exercises

1.1 Given the following vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Prove the following vector identities

(a.) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$
(b.) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$
(c.) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$

1.2 Prove the $e - \delta$ relation (1-19).

1.3 Given the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ with the Cartesian components

$$[\bar{a}^j] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad , \quad [\bar{b}^j] = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad , \quad [\bar{c}^j] = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad , \quad [\bar{d}^j] = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$$

Calculate the dyadic scalar products $\mathbf{ab} : \mathbf{cd}$ and $\mathbf{ab} \cdot\cdot\, \mathbf{cd}$.

1.4 Prove the relations (1-37) and (1-40).

1.5 Given the Cartesian components $\bar{C}^{rstu}$ of the fourth order tensor $\mathbf{C}$. Calculate the mixed curvilinear components $C_j{}^k{}_l{}^m$.

1.6 Prove the last three relations in (1-67).

1.7 Prove (1-73) by applying (1-69) to the mixed representation $\mathbf{g} = \delta_j^k\,\mathbf{g}^j\mathbf{g}_k$ of the identity tensor.

1.8 Determine the geodesics on a cylindrical surface in the three dimensional Euclidean space with arbitrary directrix.

1.9 Prove (1-97).

CHAPTER 2
Differential Theory of Surfaces

2.1 Differential geometry of surfaces, first fundamental form

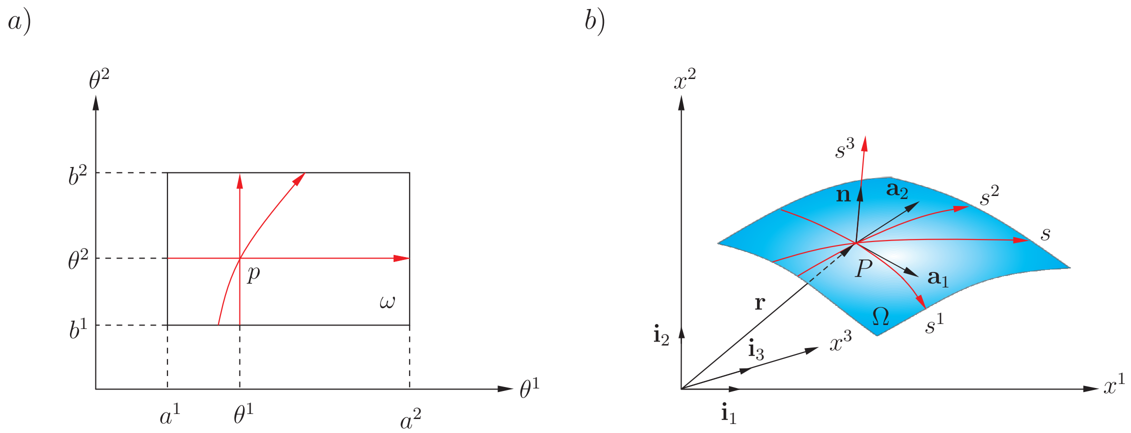

Fig. 2–1 a) Parameter space. b) Surface space.

The following section concerns surfaces in the three dimensional Euclidean space.

Let the spherical θ^3 coordinate be fixed at the value $\theta^3 = r$. Then, the mapping (1-1) of the spherical coordinates onto the Cartesian coordinates takes the form:

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} r\sin\theta^1\cos\theta^2 \\ r\sin\theta^1\sin\theta^2 \\ r\cos\theta^1 \end{bmatrix} \tag{2–1}$$

(x^1, x^2, x^3) denotes the Cartesian components of the position vector $\mathbf{x}$ to a given point of the surface. Obviously, $(x^1)^2 + (x^2)^2 + (x^3)^2 = r^2$. Hence, with the zenith angle varied in the interval $\theta^1 \in [0, \pi]$, and the azimuth angle varied in the interval $\theta^2 \in]0, 2\pi]$, (2-1) represents the parametric description of a sphere with the radius r and the center at $(x^1, x^2, x^3) = (0, 0, 0)$. In what follows it is assumed that the parametric description of all considered surfaces is defined by a constant value of the curvilinear coordinate $\theta^3 = c$ in the mapping (1-2). Then, a given surface Ω is given as, cf. (1-5):

$$x^j = f^j(\theta^1, \theta^2, c) = f^j(\theta^\alpha) \tag{2–2}$$

In the last statement of (2-2) the explicit dependence on the constant c is ignored, as will also be the case in the following. Let the mapping (2-2) be defined within a domain ω in the parameter space. For each point $p \in \omega$ determined by the parameters (θ^1, θ^2), a given point P is defined on Ω with Cartesian coordinates given by (2-2).

Assume that a curve through p is specified by the parametric description $\big(\theta^1(t), \theta^2(t)\big)$, where t is the free parameter. Then, this curve is mapped onto a curve $s = s(t)$ through P on Ω as shown on Fig. 2-1. Especially, if the curvilinear coordinate θ^2 is fixed, whereas θ^1 is varied independently a curve s^1 through P is defined on Ω. Similarly, a curve s^2 on the surface through P is obtained, if θ^1 is fixed and θ^2 is varied. The positive direction of s^1 and s^2 are defined, so positive increments of θ^α correspond to positive increments of s^α. Then, these curves define a local two-dimensional arch length coordinate system (s^1, s^2) through P.

Assume that ω is a rectangular domain $[a^1, a^2] \times [b^1, b^2]$ with sides parallel to the θ^α axes as shown on Fig. 2-1a. As an example this is the case for the mapping (2-1), where $\omega = [0, \pi] \times]0, 2\pi]$. In such cases the surface Ω will be bounded by the arc length coordinate curves given by the parameter descriptions $x^j = f^j(a^1, \theta^2)$, $x^j = f^j(a^2, \theta^2)$, $x^j = f^j(\theta^1, b^1)$ and $x^j = f^j(\theta^1, b^2)$, see Fig. 2-1b.

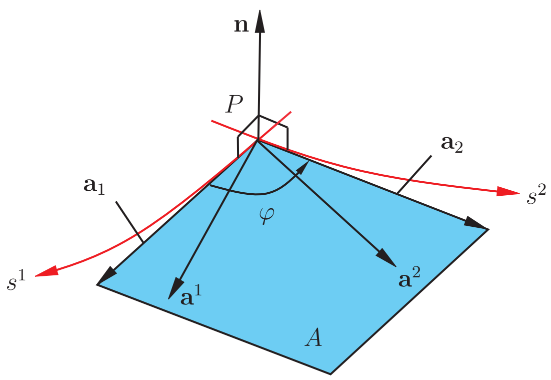

Fig. 2–2 Covariant and contravariant base vectors and surface normal unit vector.

Similar to (1-8) a covariant vector base $(\mathbf{a}_1, \mathbf{a}_2)$ may be defined at each point of the surface via the relation:

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{x}}{\partial \theta^\alpha} \quad , \quad \alpha = 1, 2 \tag{2–3}$$

Obviously, $\mathbf{a}_\alpha$ are tangential to the arch length curves at P, see Fig. 2-1b. Then, $\mathbf{a}_\alpha$ may be interpreted as a local two-dimensional covariant vector bases, which spans the tangent plane at the point P, see Fig. 2-2.

At the point P the *unit normal vector* $\mathbf{n}$ to the surface and the tangent plane is defined as:

$$\mathbf{n} = \mathbf{n}(\theta^1, \theta^2) \;=\; \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \;=\; \frac{1}{A}\,\mathbf{a}_1 \times \mathbf{a}_2 \tag{2–4}$$

A denotes the area of the parallelogram spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, and given as:

$$A \;=\; |\mathbf{a}_1 \times \mathbf{a}_2| \;=\; |\mathbf{a}_1||\mathbf{a}_2|\sin\varphi \tag{2–5}$$

The related contravariant base vectors follows from (1-16). In the present case $V = A \cdot 1 = A$. Then:

$$\left.\begin{aligned} \mathbf{a}^1 \;&=\; \frac{1}{A}\,\mathbf{a}_2 \times \mathbf{n} \;=\; \frac{1}{A^2}\;\mathbf{a}_2 \times (\mathbf{a}_1 \times \mathbf{a}_2) \;=\; \frac{1}{A^2}\Big(|\mathbf{a}_2|^2\,\mathbf{a}_1 - (\mathbf{a}_1 \cdot \mathbf{a}_2)\,\mathbf{a}_2\Big) \\ \mathbf{a}^2 \;&=\; \frac{1}{A}\,\mathbf{n} \times \mathbf{a}_1 \;=\; \frac{1}{A^2}\;(\mathbf{a}_1 \times \mathbf{a}_2) \times \mathbf{a}_1 \;=\; \frac{1}{A^2}\Big(|\mathbf{a}_1|^2\,\mathbf{a}_2 - (\mathbf{a}_1 \cdot \mathbf{a}_2)\,\mathbf{a}_1\Big) \end{aligned}\right\} \tag{2–6}$$

The last statements in (2-6) follow from the vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\,\mathbf{c}$, cf. Exercise 1.1.

It is easily shown that the covariant the contravariant base vectors in the tangent plane as given by (2-6) fulfill the orthonormality relation:

$$\mathbf{a}_\alpha \cdot \mathbf{a}^\beta \;=\; \delta_\alpha^\beta \tag{2–7}$$

where δ_α^β indicates the Kronecker's delta in two dimensions.

A *surface vector function* $\mathbf{v} \;=\; \mathbf{v}(\theta^1, \theta^2)$ is a vector field, which everywhere (i.e. for any parameters (θ^1, θ^2)) is tangential to the surface. Then, $\mathbf{v}$ may be represented by the following Cartesian, covariant and contravariant representations, cf. (1-20):

$$\mathbf{v} \;=\; \bar{v}^\alpha\,\mathbf{i}_\alpha \;=\; v^\alpha\,\mathbf{a}_\alpha \;=\; v_\alpha\,\mathbf{a}^\alpha \tag{2–8}$$

$\mathbf{i}_\alpha = \mathbf{i}^\alpha$ indicates a Cartesian vector base in the tangential plane, and $\bar{v}^j$, v^α and v_α denote the Cartesian, the covariant and the contravariant components of $\mathbf{v}$.

The Cartesian and the covariant components are related as, cf. (1-25):

$$\left.\begin{aligned} \bar{v}^\alpha \;&=\; c^\alpha_\beta\, v^\beta \quad , \quad c^\alpha_\beta \;=\; \mathbf{i}^\alpha \cdot \mathbf{a}_\beta \\ v^\alpha \;&=\; d^\alpha_\beta\, \bar{v}^\beta \quad , \quad d^\alpha_\beta \;=\; \mathbf{i}_\beta \cdot \mathbf{a}^\alpha \end{aligned}\right\} \tag{2–9}$$

where the orthogonality conditions $\mathbf{i}_\alpha \cdot \mathbf{i}^\beta = \delta_\alpha^\beta$ and $\mathbf{a}_\alpha \cdot \mathbf{a}^\beta = \delta_\alpha^\beta$ have been used.

Similarly, the covariant and contravariant components are related by the following relations:

$$\left.\begin{aligned} v_\alpha \;&=\; a_{\alpha\beta}\, v^\beta \\ v^\alpha \;&=\; a^{\alpha\beta}\, v_\beta \end{aligned}\right\} \tag{2–10}$$

where:

$$\left.\begin{aligned} a_{\alpha\beta} &= \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \\ a^{\alpha\beta} &= \mathbf{a}^\alpha \cdot \mathbf{a}^\beta \end{aligned}\right\} \tag{2–11}$$

Both $\mathbf{a}_\alpha$ and $\mathbf{a}^\alpha$ are surface vectors. Then, these can be expanded in two-dimensional contravariant and covariant vector bases as follows, cf. (1-24):

$$\left.\begin{aligned} \mathbf{a}_\alpha &= a_{\alpha\beta}\,\mathbf{a}^\beta \\ \mathbf{a}^\alpha &= a^{\alpha\beta}\,\mathbf{a}_\beta \end{aligned}\right\} \tag{2–12}$$

(2-12) is proved by scalar multiplication with $\mathbf{a}_\gamma$ and $\mathbf{a}^\gamma$ on both sides of the equation and use of (2-7) and (2-11).

Scalar multiplication on both sides of the first equation (2-12) with $\mathbf{a}^\gamma$ and use of (2-7) provides:

$$a_{\alpha\beta}\,a^{\beta\gamma} = \delta_\alpha^\gamma \tag{2–13}$$

In analogy to (1-27) a *surface second order tensor* is defined as a second order tensor, which everywhere maps a surface vector $\mathbf{v}$ onto another surface vector $\mathbf{u}$ by means of a scalar product. A second order surface tensor $\mathbf{T}$ admits the following Cartesian, covariant, contravariant and mixed representations, cf. (1-35):

$$\mathbf{T} = \bar{T}^{jk}\,\mathbf{i}_j\mathbf{i}_k = T^{\alpha\beta}\,\mathbf{a}_\alpha\mathbf{a}_\beta = T_{\alpha\beta}\,\mathbf{a}^\alpha\mathbf{a}^\beta = T^\alpha{}_\beta\,\mathbf{a}_\alpha\mathbf{a}^\beta = T_\alpha{}^\beta\,\mathbf{a}^\alpha\mathbf{a}_\beta \tag{2–14}$$

$T^{\alpha\beta}$, $T_{\alpha\beta}$, $T^\alpha{}_\beta$ and $T_\alpha{}^\beta$ denotes the covariant, the contravariant and the mixed covariant and contravariant components of the surface tensor. These are related as, cf. (1-39), (1-40):

$$\left.\begin{aligned} T_{\alpha\beta} &= a_{\alpha\gamma}a_{\beta\delta}\,T^{\gamma\delta} \quad , & T^\alpha{}_\beta &= a^{\alpha\gamma}\,T_{\gamma\beta} \\ T^{\alpha\beta} &= a^{\alpha\gamma}a^{\beta\delta}\,T_{\gamma\delta} \quad , & T_\alpha{}^\beta &= a_{\alpha\gamma}\,T^{\gamma\beta} \\ T_{\alpha\beta} &= a_{\alpha\gamma}\,T^\gamma{}_\beta \quad , & T^{\alpha\beta} &= a^{\alpha\gamma}\,T_\gamma{}^\beta \end{aligned}\right\} \tag{2–15}$$

The relations in (2-15) follows by insertion of the expansions (2-12) in (2-14).

The *surface identity tensor* $\mathbf{a}$ is defined as a second order tensor with the covariant components $a^{\alpha\beta}$, the contravariant components $a_{\alpha\beta}$, and the mixed components δ_α^β, corresponding to the following representations, cf. (1-42):

$$\mathbf{a} = a^{\alpha\beta}\,\mathbf{a}_\alpha\mathbf{a}_\beta = a_{\alpha\beta}\,\mathbf{a}^\alpha\mathbf{a}^\beta = \delta_\alpha^\beta\,\mathbf{a}^\alpha\mathbf{a}_\beta \tag{2–16}$$

The tensor $\mathbf{a}$ maps any surface vector onto itself, i.e.:

$$\mathbf{a}\cdot\mathbf{v} = \mathbf{v} \tag{2–17}$$

(2-17) is proved in the same way as (1-44), using the representation (2-16).

Let $d\mathbf{r}$ be a differential increment of the position vector $\mathbf{r}$ due to an increment $d\theta^\alpha$ of the parameters. Obviously, $d\mathbf{r}$ is a surface vector with the covariant coordinates $d\theta^\alpha$, cf. (2-3). The length ds of the incremental vector becomes:

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = d\mathbf{r} \cdot \mathbf{a} \cdot d\mathbf{r} = d\theta^\alpha \, \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \, d\theta^\beta = a_{\alpha\beta} \, d\theta^\alpha d\theta^\beta \tag{2–18}$$

The surface identity tensor $\mathbf{a}$ determines the length of any surface vector, for which reason the alternative naming *surface metric tensor* is used. The quadratic form in the last statement of (2-18) is called the *first fundamental form* of the surface.

Let $\mathbf{a}_3 = \mathbf{a}^3 = \mathbf{n}$. Then, the partial derivatives of the covariant and contravariant base vectors are unchanged given by (1-57):

$$\left.\begin{aligned} \frac{\partial \mathbf{a}_j}{\partial \theta^\alpha} &= \left\{ \begin{matrix} l \\ j \;\; \alpha \end{matrix} \right\} \mathbf{a}_l &&= \left\{ \begin{matrix} \beta \\ j \;\; \alpha \end{matrix} \right\} \mathbf{a}_\beta + \left\{ \begin{matrix} 3 \\ j \;\; \alpha \end{matrix} \right\} \mathbf{n} \\ \frac{\partial \mathbf{a}^j}{\partial \theta^\alpha} &= -\left\{ \begin{matrix} j \\ l \;\; \alpha \end{matrix} \right\} \mathbf{a}^l &&= -\left\{ \begin{matrix} j \\ \beta \;\; \alpha \end{matrix} \right\} \mathbf{a}^\beta - \left\{ \begin{matrix} j \\ 3 \;\; \alpha \end{matrix} \right\} \mathbf{n} \end{aligned}\right\} \tag{2–19}$$

$\mathbf{n}$ is orthogonal to both $\mathbf{a}_\beta$ and $\mathbf{a}^\beta$, so $\mathbf{n} \cdot \mathbf{a}_\beta = 0$ and $\mathbf{n} \cdot \mathbf{a}^\beta = 0$. Then, scalar multiplication of the first equation in (2-19) with $\mathbf{n}$ provides:

$$\left.\begin{aligned} \mathbf{n} \cdot \frac{\partial \mathbf{a}_j}{\partial \theta^\alpha} &= \left\{ \begin{matrix} 3 \\ j \;\; \alpha \end{matrix} \right\} \\ \mathbf{n} \cdot \frac{\partial \mathbf{a}^j}{\partial \theta^\alpha} &= -\left\{ \begin{matrix} j \\ 3 \;\; \alpha \end{matrix} \right\} \end{aligned}\right\} \tag{2–20}$$

Using $\mathbf{a}_3 = \mathbf{a}^3 = \mathbf{n}$, the left hand sides of (2-20) are identical for $j = 3$. However, the right hand sides have opposite signs. This can only be truth if the right hand sides vanish, leading to the following result for the Christoffel symbol:

$$\left\{ \begin{matrix} 3 \\ 3 \;\; \alpha \end{matrix} \right\} = 0 \tag{2–21}$$

In turn, (2-20) reduces to:

$$\mathbf{n} \cdot \frac{\partial \mathbf{n}}{\partial \theta^\alpha} = 0 \tag{2–22}$$

(2-22) indicates that $\frac{\partial \mathbf{n}}{\partial \theta^\alpha}$ is orthogonal to the surface normal $\mathbf{n}$, and hence must be a surface vector.

Obviously, the differential increment vector $d\boldsymbol{\theta} = d\theta^\beta \, \mathbf{a}_\beta$ with the covariant components $d\theta^\beta$ is a surface vector. The gradient $\boldsymbol{\nabla}\mathbf{v}$ of a surface vector function $\mathbf{v} = \mathbf{v}(\theta^\gamma)$ is a surface second order tensor, which associates differential increment vector $d\boldsymbol{\theta}$ to a surface differential increment vector $d\mathbf{v} = \boldsymbol{\nabla}\mathbf{v} \cdot d\boldsymbol{\theta} = \frac{\partial \mathbf{v}}{\partial \theta^\alpha} \, d\theta^\alpha$ of the vector $\mathbf{v}$. The gradient tensor has the following curvilinear representations, which merely follow by replacing the Latin indices in (1-64), (1-65) with Greek indices:

$$\boldsymbol{\nabla}\mathbf{v} \;=\; v^{\alpha}{}_{;\beta}\,\mathbf{a}_{\alpha}\mathbf{a}^{\beta} \;=\; v_{\alpha;\beta}\,\mathbf{a}^{\alpha}\mathbf{a}^{\beta} \tag{2–23}$$

where:

$$\left.\begin{aligned} v^{\alpha}{}_{;\beta} \;&=\; \frac{\partial v^{\alpha}}{\partial\theta^{\beta}} + \begin{Bmatrix}\alpha\\ \beta\;\;\gamma\end{Bmatrix} v^{\gamma} \\ v_{\alpha;\beta} \;&=\; \frac{\partial v_{\alpha}}{\partial\theta^{\beta}} - \begin{Bmatrix}\gamma\\ \alpha\;\;\beta\end{Bmatrix} v_{\gamma} \end{aligned}\right\} \tag{2–24}$$

$v^{\alpha}{}_{;\beta}$ and $v_{\alpha;\beta}$ are referred to as the *surface contravariant derivative* and *surface covariant derivative* of the components v_{α} and v^{α}.

Correspondingly, the gradient $\boldsymbol{\nabla}\mathbf{T}$ of a surface second order tensor function $\mathbf{T} = \mathbf{T}(\theta^{\gamma})$ is a surface third order tensor with the following curvilinear representations, cf. (1-69), (1-70):

$$\boldsymbol{\nabla}\mathbf{T} \;=\; T^{\alpha\beta}{}_{;\gamma}\,\mathbf{a}_{\alpha}\mathbf{a}_{\beta}\mathbf{a}^{\gamma} \;=\; T^{\alpha}{}_{;\beta\gamma}\,\mathbf{a}_{\alpha}\mathbf{a}^{\beta}\mathbf{a}^{\gamma} \;=\; T_{\alpha}{}^{\beta}{}_{;\gamma}\,\mathbf{a}^{\alpha}\mathbf{a}_{\beta}\mathbf{a}^{\gamma} \;=\; T_{\alpha\beta;\gamma}\,\mathbf{a}^{\alpha}\mathbf{a}^{\beta}\mathbf{a}^{\gamma} \tag{2–25}$$

where:

$$\left.\begin{aligned} T^{\alpha\beta}{}_{;\gamma} \;&=\; \frac{\partial T^{\alpha\beta}}{\partial\theta^{\gamma}} + T^{\delta\beta}\begin{Bmatrix}\alpha\\ \delta\;\;\gamma\end{Bmatrix} + T^{\alpha\delta}\begin{Bmatrix}\beta\\ \delta\;\;\gamma\end{Bmatrix} \\ T^{\alpha}{}_{\beta;\gamma} \;&=\; \frac{\partial T^{\alpha}{}_{\beta}}{\partial\theta^{\gamma}} + T^{\delta}{}_{\beta}\begin{Bmatrix}\alpha\\ \delta\;\;\gamma\end{Bmatrix} - T^{\alpha}{}_{\delta}\begin{Bmatrix}\delta\\ \beta\;\;\gamma\end{Bmatrix} \\ T_{\alpha}{}^{\beta}{}_{;\gamma} \;&=\; \frac{\partial T_{\alpha}{}^{\beta}}{\partial\theta^{\gamma}} - T_{\delta}{}^{\beta}\begin{Bmatrix}\delta\\ \alpha\;\;\gamma\end{Bmatrix} + T_{\alpha}{}^{\delta}\begin{Bmatrix}\beta\\ \delta\;\;\gamma\end{Bmatrix} \\ T_{\alpha\beta;\gamma} \;&=\; \frac{\partial T_{\alpha\beta}}{\partial\theta^{\gamma}} - T_{\beta\delta}\begin{Bmatrix}\delta\\ \alpha\;\;\gamma\end{Bmatrix} - T_{\alpha\delta}\begin{Bmatrix}\delta\\ \beta\;\;\gamma\end{Bmatrix} \end{aligned}\right\} \tag{2–26}$$

The gradient of the gradient of a surface vector is a third order surface tensor with the following representations, cf. (1-74), (1-75), (1-76):

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{v}) \;=\; v^{\alpha}{}_{;\beta\gamma}\,\mathbf{a}_{\alpha}\mathbf{a}^{\beta}\mathbf{a}^{\gamma} \;=\; v_{\alpha;\beta\gamma}\,\mathbf{a}^{\alpha}\mathbf{a}^{\beta}\mathbf{a}^{\gamma} \tag{2–27}$$

where:

$$\begin{aligned} v^{\alpha}{}_{;\beta\gamma} \;=\; (v^{\alpha}{}_{;\beta})_{;\gamma} \;=\; &\frac{\partial^2 v^{\alpha}}{\partial\theta^{\beta}\partial\theta^{\gamma}} + \begin{Bmatrix}\alpha\\ \beta\;\;\mu\end{Bmatrix}\frac{\partial v^{\mu}}{\partial\theta^{\gamma}} + \begin{Bmatrix}\alpha\\ \gamma\;\;\mu\end{Bmatrix}\frac{\partial v^{\mu}}{\partial\theta^{\beta}} - \begin{Bmatrix}\mu\\ \beta\;\;\gamma\end{Bmatrix}\frac{\partial v^{\alpha}}{\partial\theta^{\mu}} + \\ &\frac{\partial}{\partial\theta^{\gamma}}\begin{Bmatrix}\alpha\\ \beta\;\;\mu\end{Bmatrix} v^{\mu} + \begin{Bmatrix}\alpha\\ \gamma\;\;\nu\end{Bmatrix}\begin{Bmatrix}\nu\\ \beta\;\;\mu\end{Bmatrix} v^{\mu} - \begin{Bmatrix}\alpha\\ \nu\;\;\mu\end{Bmatrix}\begin{Bmatrix}\nu\\ \beta\;\;\gamma\end{Bmatrix} v^{\mu} \end{aligned} \tag{2–28}$$

$$\begin{aligned} v_{\alpha;\beta\gamma} \;=\; (v_{\alpha;\beta})_{;\gamma} \;=\; &\frac{\partial^2 v_{\alpha}}{\partial\theta^{\beta}\partial\theta^{\gamma}} - \begin{Bmatrix}\mu\\ \alpha\;\;\beta\end{Bmatrix}\frac{\partial v_{\mu}}{\partial\theta^{\gamma}} - \begin{Bmatrix}\mu\\ \alpha\;\;\gamma\end{Bmatrix}\frac{\partial v_{\mu}}{\partial\theta^{\beta}} - \begin{Bmatrix}\mu\\ \beta\;\;\gamma\end{Bmatrix}\frac{\partial v_{\alpha}}{\partial\theta^{\mu}} - \\ &\frac{\partial}{\partial\theta^{\gamma}}\begin{Bmatrix}\mu\\ \alpha\;\;\beta\end{Bmatrix} v_{\mu} + \begin{Bmatrix}\nu\\ \alpha\;\;\gamma\end{Bmatrix}\begin{Bmatrix}\mu\\ \nu\;\;\beta\end{Bmatrix} v_{\mu} + \begin{Bmatrix}\nu\\ \beta\;\;\gamma\end{Bmatrix}\begin{Bmatrix}\mu\\ \nu\;\;\alpha\end{Bmatrix} v_{\mu} \end{aligned} \tag{2–29}$$

The *surface Riemann-Christoffel tensor* is denoted $\mathbf{B} = \mathbf{B}(\theta^1, \theta^2) = B^{\delta}{}_{\alpha\beta\gamma}\, \mathbf{a}_\delta \mathbf{a}^\alpha \mathbf{a}^\beta \mathbf{a}^\gamma$ to distinguish it from the equivalent tensor $\mathbf{R}$ in the three-dimensional Riemann space. The mixed covariant and contravariant components are given as, cf. (1-78):

$$B^{\delta}{}_{\alpha\beta\gamma} = \left\{ \begin{matrix} \nu \\ \alpha \;\; \gamma \end{matrix} \right\} \left\{ \begin{matrix} \delta \\ \nu \;\; \beta \end{matrix} \right\} - \left\{ \begin{matrix} \nu \\ \alpha \;\; \beta \end{matrix} \right\} \left\{ \begin{matrix} \delta \\ \nu \;\; \gamma \end{matrix} \right\} + \frac{\partial}{\partial \theta^\beta} \left\{ \begin{matrix} \delta \\ \alpha \;\; \gamma \end{matrix} \right\} - \frac{\partial}{\partial \theta^\gamma} \left\{ \begin{matrix} \delta \\ \alpha \;\; \beta \end{matrix} \right\} \tag{2–30}$$

In analogy to (1-77), (1-79) the tensor components $B^{\delta}{}_{\alpha\beta\gamma}$ fulfill:

$$\left. \begin{aligned} B^{\delta}{}_{\alpha\beta\gamma} &= -B^{\delta}{}_{\alpha\gamma\beta} \\ B^{\delta}{}_{\alpha\beta\gamma}\, v_\delta &= v_{\alpha;\beta\gamma} - v_{\alpha;\gamma\beta} \end{aligned} \right\} \tag{2–31}$$

Further, Bianchi's identities in (1-80), (1-98) attain the form:

$$\left. \begin{aligned} & B^{\mu}{}_{\alpha\beta\gamma} + B^{\mu}{}_{\beta\gamma\alpha} + B^{\mu}{}_{\gamma\alpha\beta} = 0 \\ & B^{\mu}{}_{\alpha\beta\gamma;\delta} + B^{\mu}{}_{\alpha\gamma\delta;\beta} + B^{\mu}{}_{\alpha\delta\beta;\gamma} = 0 \end{aligned} \right\} \tag{2–32}$$

From (1-78) and (2-32) follow that the components in the tangent plane $R^{\delta}{}_{\alpha\beta\gamma}$ of the Riemann-Christoffel tensor in the three-dimensional space is related to the corresponding components $B^{\delta}{}_{\alpha\beta\gamma}$ of the surface Riemann-Christoffel tensor as follows:

$$R^{\delta}{}_{\alpha\beta\gamma} = B^{\delta}{}_{\alpha\beta\gamma} + \left\{ \begin{matrix} 3 \\ \alpha \;\; \gamma \end{matrix} \right\} \left\{ \begin{matrix} \delta \\ 3 \;\; \beta \end{matrix} \right\} - \left\{ \begin{matrix} 3 \\ \alpha \;\; \beta \end{matrix} \right\} \left\{ \begin{matrix} \delta \\ 3 \;\; \gamma \end{matrix} \right\} \tag{2–33}$$

Since, the three-dimensional Euclidian space is flat, we have everywhere $R^{\delta}{}_{\alpha\beta\gamma} = 0$. Then, the following simplified representation for tensor components $B^{\delta}{}_{\alpha\beta\gamma}$ is obtained from (2-33):

$$B^{\delta}{}_{\alpha\beta\gamma} = \left\{ \begin{matrix} 3 \\ \alpha \;\; \beta \end{matrix} \right\} \left\{ \begin{matrix} \delta \\ 3 \;\; \gamma \end{matrix} \right\} - \left\{ \begin{matrix} 3 \\ \alpha \;\; \gamma \end{matrix} \right\} \left\{ \begin{matrix} \delta \\ 3 \;\; \beta \end{matrix} \right\} \tag{2–34}$$

Finally, contraction of the indices α and δ in (2-34) provides:

$$B^{\alpha}{}_{\alpha\beta\gamma} = 0 \tag{2–35}$$

Due to the symmetry condition in (2-31) and the first Bianchi's identity (2-32) only one independent and non-trivial tensor component exists, which is taken as $B^{1}{}_{212}$.

2.2 Principal curvatures, second fundamental form

An arbitrary curve $s(t)$ on the surface Ω is defined by the parametric description $\mathbf{r} = \mathbf{r}(t) = \mathbf{r}\big(\theta^1(t), \theta^2(t)\big)$, where t is a free parameter as explained subsequent to (2-2). P and Q denote two neighboring points on the curve defined by the position vectors $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$, corresponding to the parameter values t and $t + dt$, respectively. The increment $d\mathbf{r}$ with the length ds is a surface vector placed in the tangent plane at P, and specified by the covariant representation $d\mathbf{r} = d\theta^\alpha\, \mathbf{a}_\alpha$. Hence, the unit tangential vector to the curve at P is given as:

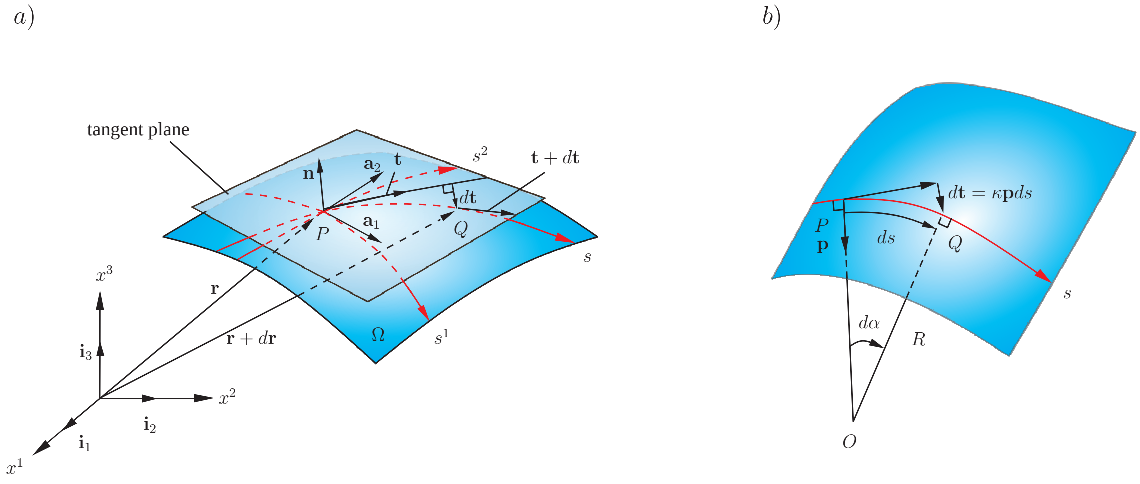

Fig. 2–3 a) Unit tangent vector of a curve. b) Radius of curvature of a curve.

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\theta^\alpha}{ds}\mathbf{a}_\alpha \tag{2–36}$$

The unit tangential vector in Q deviates infinitesimally from $\mathbf{t}$, and may be written as $\mathbf{t}+d\mathbf{t}$. Given, that the length unchanged is equal to 1, the increment $d\mathbf{t}$ must fulfill:

$$\begin{aligned} &1 = (\mathbf{t}+d\mathbf{t})\cdot(\mathbf{t}+d\mathbf{t}) = \mathbf{t}\cdot\mathbf{t} + 2\,d\mathbf{t}\cdot\mathbf{t} + d\mathbf{t}\cdot d\mathbf{t} = 1 + 2\,d\mathbf{t}\cdot\mathbf{t} \quad \Rightarrow \\ &d\mathbf{t}\cdot\mathbf{t} = 0 \end{aligned} \tag{2–37}$$

where the second order term $d\mathbf{t}\cdot d\mathbf{t}$ is ignored. (2-37) shows that the increment $d\mathbf{t}$ is orthogonal to $\mathbf{t}$. Let $\mathbf{p}$ denote the unit normal vector co-directional to $d\mathbf{t}$. Then, the following identity may be defined:

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{p} \tag{2–38}$$

(2-38) is referred to as *Frenet's formula*. $\frac{d\mathbf{t}}{ds}$ and $\mathbf{p}$ are called the *curvature vector* and the *principal normal vector* of the curve, respectively, and the proportionality factor κ is referred to as the *curvature*. Generally, this is always assumed to be positive, so (2-38) is defining the orientation of $\mathbf{p}$.

In Fig. 2-3b lines orthogonal to s have been drawn through P and Q in the plane spanned by $\mathbf{t}$ and $\mathbf{p}$, which intersect each other at the *curvature center* O under the angle $d\alpha$ as shown on Fig. 2-3b. The position of O relative to s is defined by the orientation of $\mathbf{p}$. Then the following geometrical relation prevails:

$$d\alpha = \frac{ds}{R} \tag{2–39}$$

where $R = OP = OQ$ denotes *the radius of curvature* at P.

From the similar triangles in Fig. 2-3b follows that $d\alpha = \kappa ds$. Then, (1-39) provides the following geometrical interpretation of the curvature:

$$\kappa = \frac{1}{R} \tag{2–40}$$

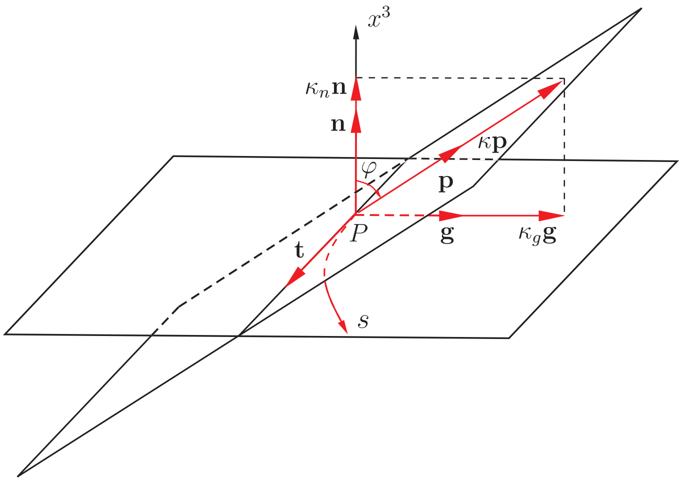

Fig. 2–4 Definition of principal curvature κ, normal curvature κ_n and geodesic curvature κ_g.

Still another unit vector $\mathbf{g}$, placed in the tangential plane and orthogonal to $\mathbf{t}$, may be considered, see Fig. 2-4. The unit vectors $\mathbf{n}$, $\mathbf{p}$ and $\mathbf{g}$ are all placed in a plane orthogonal to $\mathbf{t}$, and $\mathbf{n}$ and $\mathbf{g}$ are orthogonal to each other. Hence, $\kappa\,\mathbf{p}$ may be represented as a linear combination of $\mathbf{n}$ and $\mathbf{g}$:

$$\kappa\,\mathbf{p} = \kappa_n\,\mathbf{n} + \kappa_g\,\mathbf{g} \tag{2–41}$$

The expansion components κ_n and κ_g are known as the *normal curvature* and the *geodesic curvature* of the curve. Using $\mathbf{n}\cdot\mathbf{g} = 0$, it follows from (2-38) and (2-41) that these may be expressed in terms of the principal curvature as:

$$\left.\begin{aligned} \kappa_n &= \mathbf{n}\cdot\mathbf{p}\,\kappa = \cos\varphi\,\kappa = \mathbf{n}\cdot\frac{d\mathbf{t}}{ds} \\ \kappa_g &= \mathbf{g}\cdot\mathbf{p}\,\kappa = \sin\varphi\,\kappa = \mathbf{g}\cdot\frac{d\mathbf{t}}{ds} \end{aligned}\right\} \tag{2–42}$$

where φ indicates the angle between $\mathbf{n}$ and $\mathbf{g}$, see Fig. 2-4.

From (2-19) and (2-36) follows that the curvature vector $\frac{d\mathbf{t}}{ds}$ has the following covariant representation:

$$\begin{aligned} &\frac{d\mathbf{t}}{ds} = \frac{d^2\theta^\gamma}{ds^2}\mathbf{a}_\gamma + \frac{\partial \mathbf{a}_\alpha}{\partial\theta^\beta}\frac{d\theta^\alpha}{ds}\frac{d\theta^\beta}{ds} = \\ &\frac{d^2\theta^\gamma}{ds^2}\mathbf{a}_\gamma + \left(\left\{\begin{matrix}\gamma\\ \alpha\ \ \beta\end{matrix}\right\}\mathbf{a}_\gamma + \left\{\begin{matrix}3\\ \alpha\ \ \beta\end{matrix}\right\}\mathbf{n}\right)\frac{d\theta^\alpha}{ds}\frac{d\theta^\beta}{ds} = \\ &\left(\frac{d^2\theta^\gamma}{ds^2} + \left\{\begin{matrix}\gamma\\ \alpha\ \ \beta\end{matrix}\right\}\frac{d\theta^\alpha}{ds}\frac{d\theta^\beta}{ds}\right)\mathbf{a}_\gamma + \left\{\begin{matrix}3\\ \alpha\ \ \beta\end{matrix}\right\}\frac{d\theta^\alpha}{ds}\frac{d\theta^\beta}{ds}\mathbf{n} \end{aligned} \tag{2–43}$$

If the surface curve is a geodesic in the two-dimension Riemann space made up of the considered surface, the first term in the last statement of (2-43) vanish along the curve, cf. (1-90).

Since $\mathbf{n} \cdot \mathbf{g} = 0$ it follows from (2-42) and (2-43) follows that the geodesic curvature of a geodesic curve becomes:

$$\kappa_g = \begin{Bmatrix} 3 \\ \alpha \;\; \beta \end{Bmatrix} \frac{d\theta^\alpha}{ds} \frac{d\theta^\beta}{ds} \mathbf{n} \cdot \mathbf{g} = 0 \tag{2–44}$$

In turn, this means that the surface normal $\mathbf{n}$ and the principal normal vector $\mathbf{p}$ of a geodesic are coincident so $\kappa = \kappa_n$, cf. (1-41).

Since $\mathbf{n} \cdot \mathbf{a}_\gamma = 0$, it follows from (2-42) and (2-43) that the normal curvature of an arbitrary curve on the surface can be written as:

$$\kappa_n = \mathbf{n} \cdot \frac{d\mathbf{t}}{ds} = \begin{Bmatrix} 3 \\ \alpha \;\; \beta \end{Bmatrix} \frac{d\theta^\alpha}{ds} \frac{d\theta^\beta}{ds} = b_{\alpha\beta} \frac{d\theta^\alpha}{ds} \frac{d\theta^\beta}{ds} \tag{2–45}$$

where the following indexed quantity has been introduced:

$$b_{\alpha\beta} = \begin{Bmatrix} 3 \\ \alpha \;\; \beta \end{Bmatrix} = b_{\beta\alpha} \tag{2–46}$$

Two neighbouring points P and Q on the curve s are given by the position vectors $\mathbf{r}(s)$ and $\mathbf{r}(s + ds) = \mathbf{r}(s) + d\mathbf{r}$. By the use of (2-36) the increment $d\mathbf{r}$ may be represented by the Taylor expansion:

$$d\mathbf{r} = \mathbf{r}(s+ds) - \mathbf{r}(s) = \frac{d\mathbf{r}}{ds} ds + \frac{1}{2} \frac{d^2\mathbf{r}}{ds^2} ds^2 + \mathbf{O}\big(ds^3\big) = \mathbf{t}\, ds + \frac{1}{2} \frac{d\mathbf{t}}{ds} ds^2 + \mathbf{O}(ds^3) \tag{2–47}$$

The distance dn from Q to the tangent plane is given by the projection of $d\mathbf{r}$ on the surface unit normal vector $\mathbf{n}$ at P:

$$dn = \mathbf{n} \cdot d\mathbf{r} = \frac{1}{2} \mathbf{n} \cdot \frac{d\mathbf{t}}{ds} ds^2 = \frac{1}{2} \kappa_n\, ds^2 = \frac{1}{2} b_{\alpha\beta}\, d\theta^\alpha d\theta^\beta \tag{2–48}$$

where (2-42) and (2-45) have been used.

The larger the distance dn, the larger is the normal curvature κ_n of the curve at P. In (2-48) this quantity is partly determined by the components $b_{\alpha\beta}$ and partly by the parameter increments $d\theta^\alpha$. As follows from (2-46) the components $b_{\alpha\beta}$ are entirely determined from properties related to the surface, and independent of the specific direction of the curve s. In contrast this direction is determined by the increments $d\theta^\alpha$ of the curvilinear coordinates.

The quadratic form $b_{\alpha\beta}\, d\theta^\alpha d\theta^\beta$ entering (2-48) is called the *second fundamental form* of the surface. In what follows $b_{\alpha\beta}$ is considered the contravariant components of a surface second order tensor $\mathbf{b} = b_{\alpha\beta}\, \mathbf{a}^\alpha \mathbf{a}^\beta$, which is called the *curvature tensor* of the surface.

Using $\mathbf{n} \cdot \mathbf{a}_\beta = \mathbf{n} \cdot \mathbf{a}^\beta = 0$, it follows from (2-19) that:

$$\begin{aligned} &\mathbf{n} \cdot \frac{\partial \mathbf{a}_\alpha}{\partial \theta_\beta} = \left\{ \begin{matrix} 3 \\ \alpha \;\; \beta \end{matrix} \right\} = b_{\alpha\beta} \quad \Rightarrow \\ &b_{\alpha\beta} = \mathbf{n} \cdot \frac{\partial \mathbf{a}_\alpha}{\partial \theta^\beta} = \mathbf{n} \cdot \frac{\partial \big(a_{\alpha\gamma} \mathbf{a}^\gamma \big)}{\partial \theta^\beta} = \mathbf{n} \cdot \mathbf{a}^\gamma \, \frac{\partial a_{\alpha\gamma}}{\partial \theta^\beta} + a_{\alpha\gamma} \, \mathbf{n} \cdot \frac{\partial \mathbf{a}^\gamma}{\partial \theta^\beta} = -a_{\alpha\gamma} \left\{ \begin{matrix} \gamma \\ 3 \;\; \beta \end{matrix} \right\} \end{aligned} \tag{2–49}$$

where the last relation in (2-19) has been used. Then, use of (2-13) provides:

$$\left\{ \begin{matrix} \alpha \\ 3 \;\; \beta \end{matrix} \right\} = -a^{\alpha\gamma} \, b_{\gamma\beta} = -b^\alpha{}_\beta \tag{2–50}$$

Insertion of (2-46) and (2-50) into (2-34) provides the following alternative representation of the components of the surface Riemann-Christoffel tensor:

$$\begin{aligned} B^\delta{}_{\alpha\beta\gamma} &= -b_{\alpha\beta} b^\delta{}_\gamma + b_{\alpha\gamma} b^\delta{}_\beta \quad \Rightarrow \\ B_{\delta\alpha\beta\gamma} &= -b_{\alpha\beta} b_{\delta\gamma} + b_{\alpha\gamma} b_{\delta\beta} \end{aligned} \tag{2–51}$$

where the transformation rules (2-15) of covariant and contravariant tensor components have been used in the final statement. (2-51) provides the following form for the selected independent tensor component B_{1212}:

$$B_{1212} = -b_{21} b_{12} + b_{22} b_{11} = \det \big[b_{\alpha\beta} \big] = b \tag{2–52}$$

where b denotes the determinant formed by the contravariant components of the curvature tensor $\mathbf{b}$.

(2-52) is called *Gauss's equation*. If $B_{1212} = 0$, all other contravariant components of the tensor will vanish as well, so it may be concluded that $\mathbf{B} = \mathbf{0}$. Then, a surface is flat (i.e. a plane), if $b = 0$ everywhere. Noticed that if the determinant of the contravariant components vanish, the determinant of the components of any other representation of $\mathbf{b}$ vanishes as well.

From (2-18) and (2-45) follows that the normal curvature of a curve on the surface with a direction determined by the increment $d\theta^\alpha$ may be written as:

$$\kappa_n = \frac{b_{\alpha\beta} \, d\theta^\alpha d\theta^\beta}{a_{\alpha\beta} \, d\theta^\alpha d\theta^\beta} \tag{2–53}$$

(2-53) may be interpreted as a *Rayleigh quotient* related to the following generalized eigenvalue problem, (Nielsen, 2006):

$$\big(b_{\alpha\beta} - \kappa_n \, a_{\alpha\beta} \big) \, t^\beta = 0 \tag{2–54}$$

$t^\beta = \frac{d\theta^\beta}{ds}$ represent the covariant components of a unit vector $\mathbf{t} = t^\beta \, \mathbf{a}_\beta$ placed in the tangential plane to the surface at P, which is an eigenvector to (2-54). In tensor format the eigenvalue problem reads:

$$\big(\mathbf{b} - \kappa_n\, \mathbf{a}\big) \cdot \mathbf{t} = \mathbf{0} \tag{2–55}$$

(2-55) is fulfilled for two linear independent eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$ related to the eigenvalues κ_1 and κ_2. The directions on the surface by the eigenvectors $\mathbf{t}_\gamma$ are denoted the *principal curvature directions*, and the related eigenvalues are called *principal curvatures*. The unit eigenvectors fulfill the orthogonality conditions, (Nielsen, 2006):

$$\mathbf{t}_\gamma \cdot \mathbf{a} \cdot \mathbf{t}_\delta \;=\; \begin{cases} 0 \quad , \quad \gamma \neq \delta \\ 1 \quad , \quad \gamma = \delta \end{cases} \tag{2–56}$$

$$\mathbf{t}_\gamma \cdot \mathbf{b} \cdot \mathbf{t}_\delta \;=\; \begin{cases} 0 \quad , \quad \gamma \neq \delta \\ \kappa_\gamma \;\, , \quad \gamma = \delta \end{cases} \tag{2–57}$$

In (2-56) it has been used that $\mathbf{a}$ is an identity tensor, so $\mathbf{t}_{(\gamma)} \cdot \mathbf{a} \cdot \mathbf{t}_{(\gamma)} = \mathbf{t}_{(\gamma)} \cdot \mathbf{t}_{(\gamma)} = 1$.

Let the principal curvatures be ordered, so κ_1 denotes the smallest and κ_2 the largest eigenvalue. As follows from the well-known bounds on the Rayleigh quotient the normal curvature κ_n of an arbitrary curve passing through P will be bounded by the principal curvatures as, (Nielsen, 2006):

$$\kappa_1 \;\leq\; \kappa_n \;\leq\; \kappa_2 \tag{2–58}$$

Using the mixed representations $\mathbf{a} = \delta^\alpha_\beta\, \mathbf{a}_\alpha \mathbf{a}^\beta$ and $\mathbf{b} = b^\alpha{}_\beta\, \mathbf{a}_\alpha \mathbf{a}^\beta$ the component form of (2-55) attains the form:

$$\big(b^\alpha{}_\beta - \kappa_n\, \delta^\alpha_\beta\big)\, t^\beta = 0 \tag{2–59}$$

Formally, (2-59) may be obtained by contraction of (2-54) with $a^{\gamma\alpha}$ and use of (2-13), (2-15). Solutions $t^\beta \neq 0$ of the homogeneous system of linear equations (1-157) are obtained, if the determinant of the coefficient matrix vanishes. This provides the following *characteristic equation* for the determination of the principal curvatures:

$$\begin{aligned} &\det\big[b^\alpha{}_\beta - \kappa\, \delta^\alpha_\beta\big] \;=\; \kappa_n^2 - \big(b^1{}_1 + b^2{}_2\big)\kappa_n + \big(b^1{}_1 b^2{}_2 - b^2{}_1 b^1{}_2\big) = 0 \qquad \Rightarrow \\ &\kappa_n^2 - b^\alpha{}_\alpha\, \kappa_n + \frac{b}{a} = 0 \end{aligned} \tag{2–60}$$

In the last statement it has been used that $\det\big[b^\gamma{}_\beta\big] \;=\; \frac{b}{a}$, where $a \;=\; \det\big[a_{\alpha\gamma}\big]$ denotes the determinant of the matrix formed by the contravariant components of the surface identity tensor. This relation follows from the identities:

$$\begin{aligned} b_{\alpha\beta} \quad &= \; a_{\alpha\gamma}\, b^\gamma{}_\beta && \Rightarrow \\ \det\big[b_{\alpha\beta}\big] \; &= \; \det\big[a_{\alpha\gamma}\big]\; \det\big[b^\gamma{}_\beta\big] && \Rightarrow \\ \det\big[b^\gamma{}_\beta\big] \; &= \; \frac{b}{a} \end{aligned} \tag{2–61}$$

The solutions to (2-60) may be written as:

$$\left.\begin{array}{l}\kappa_1\\ \kappa_2\end{array}\right\} = H \mp \sqrt{H^2 - K} \qquad (2–62)$$

where:

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} b^{\alpha}_{\ \alpha} \qquad (2–63)$$

$$K = \kappa_1 \kappa_2 = \frac{b}{a} = b^1_{\ 1} b^2_{\ 2} - b^2_{\ 1} b^1_{\ 2} \qquad (2–64)$$

H is called the *mean curvature* of the surface, and K is the *Gaussian curvature*. As seen K is equal to the determinant of the matrix formed by the mixed components $b^{\gamma}_{\ \beta}$ of the curvature tensor, and $2H$ is equal to the trace of this matrix.

It follows from that $K = 0$, if either $\kappa_1 = 0$ or $\kappa_2 = 0$. A surface, where the Gaussian curvature vanishes everywhere is developable, i.e. the surface can be constructed by transforming a plane without distortion. Conical and cylindrical surfaces are developable, whereas a spherical surface in not. From (2-52) and (2-64) follows:

$$B_{1212} = aK \qquad (2–65)$$

As follows from (2-56), $\mathbf{t}_1 \cdot \mathbf{a} \cdot \mathbf{t}_2 = \mathbf{t}_1 \cdot \mathbf{t}_2 = 0$. This means that the principal curvature directions through P are *orthogonal to each other*. It is then possible to construct an (s^1, s^2) arc-length coordinate system at each point on the surface, which everywhere has the principal curvature unit eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$ as tangents. This so-called *principal curvature coordinate system* is specific in the sense that both $[a_{\alpha\beta}]$ and $[b_{\alpha\beta}]$ become diagonal, i.e $a_{12} = b_{12} = 0$.

The principal curvatures in this specific coordinate system are given as:

$$\kappa_\alpha = \frac{b_{(\alpha)(\alpha)}}{a_{(\alpha)(\alpha)}} \qquad (2–66)$$

In the following some relations are described, which involve the partial derivative of the surface unit normal vector $\mathbf{n}$. Partial differentiation of the equation $\mathbf{n} \cdot \mathbf{a}_\alpha = 0$ and use of (2-19) provides:

$$\frac{\partial \mathbf{n}}{\partial \theta^\beta} \cdot \mathbf{a}_\alpha + \mathbf{n} \cdot \frac{\partial \mathbf{a}_\alpha}{\partial \theta^\beta} = 0 \qquad \Rightarrow$$

$$\mathbf{a}_\alpha \cdot \frac{\partial \mathbf{n}}{\partial \theta^\beta} = -\mathbf{n} \cdot \frac{\partial \mathbf{a}_\alpha}{\partial \theta^\beta} = -\left\{ \begin{array}{c} 3 \\ \alpha \ \ \beta \end{array} \right\} = -b_{\alpha\beta} \qquad (2–67)$$

$\mathbf{n}$ is a unit vector, so $\mathbf{n} \cdot \mathbf{n} = 1$. Then, partial differentiation provides an alternative derivation of (2-22):

$$\mathbf{n} \cdot \frac{\partial \mathbf{n}}{\partial \theta^\alpha} = 0 \qquad (2–68)$$

Hence, $\frac{\partial \mathbf{n}}{\partial \theta^\alpha}$ is orthogonal to $\mathbf{n}$, and accordingly can be decomposed in the covariant surface vector base $\mathbf{a}_\beta$ as follows:

$$\frac{\partial \mathbf{n}}{\partial \theta^\alpha} = l_\alpha^\beta \, \mathbf{a}_\beta \tag{2–69}$$

Scalar multiplication of both sides with $\mathbf{a}_\gamma$ and use of (2-15) and (2-67) provides the following solution for the components l_α^β:

$$\mathbf{a}_\gamma \cdot \frac{\partial \mathbf{n}}{\partial \theta^\alpha} = -b_{\gamma\alpha} = l_\alpha^\beta \, \mathbf{a}_\gamma \cdot \mathbf{a}_\beta = a_{\gamma\beta} \, l_\alpha^\beta \qquad \Rightarrow$$

$$l_\alpha^\beta = -a^{\beta\gamma} \, b_{\gamma\alpha} = -b^\beta{}_\alpha \tag{2–70}$$

Then, insertion into (2-69) provides the following alternative representations:

$$\frac{\partial \mathbf{n}}{\partial \theta^\alpha} = -b^\beta{}_\alpha \, \mathbf{a}_\beta = -b_{\gamma\alpha} \, a^{\beta\gamma} \, \mathbf{a}_\beta = -b_{\alpha\beta} \, \mathbf{a}^\beta = -\begin{Bmatrix} 3 \\ \alpha \;\; \beta \end{Bmatrix} \mathbf{a}^\beta \tag{2–71}$$

where (2-12), (2-15) and (2-46) have been used.

Similarly, (2-19) and (2-46) provide the following relation for the partial derivatives of the surface covariant base vectors:

$$\frac{\partial \mathbf{a}_\alpha}{\partial \theta^\beta} = \begin{Bmatrix} \gamma \\ \alpha \;\; \beta \end{Bmatrix} \mathbf{a}_\gamma + b_{\alpha\beta} \, \mathbf{n} \tag{2–72}$$

(2-71) is called *Gauss's formula*, not to be confused with Gauss's equation given by Eq. (2-52).

$\mathrm{e}_{\alpha\beta}$ and $\mathrm{e}^{\alpha\beta}$ indicate *two-dimensional permutation symbols* defined as:

$$\left.\begin{aligned} \big[e_{\alpha\beta}\big] &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \big[e^{\alpha\beta}\big] &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}\right\} \tag{2–73}$$

As seen $[\mathrm{e}^{\alpha\beta}] = [\mathrm{e}_{\alpha\beta}]^{-1} = [\mathrm{e}_{\alpha\beta}]^T = -[\mathrm{e}_{\alpha\beta}]$.

Define:

$$\left.\begin{aligned} a_1 &= |\mathbf{a}_1| = (\mathbf{a}_1 \cdot \mathbf{a}_1)^{1/2} = \sqrt{a_{11}} \\ a_2 &= |\mathbf{a}_2| = (\mathbf{a}_2 \cdot \mathbf{a}_2)^{1/2} = \sqrt{a_{22}} \\ \cos\varphi &= \frac{\mathbf{a}_1 \cdot \mathbf{a}_2}{|\mathbf{a}_1||\mathbf{a}_2|} = \frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}} \quad \Rightarrow \quad \sin\varphi = \sqrt{1 - \frac{a_{12}^2}{a_{11}a_{22}}} \end{aligned}\right\} \tag{2–74}$$

where the angle φ is defined in Fig. 2-2. Further, (2-11) has been used.

Then, (2-5) may be written as:

$$A = \sqrt{a_{11}}\sqrt{a_{22}}\,\sin\varphi = \sqrt{a_{11}a_{22}-(a_{12})^2} = \sqrt{a} \tag{2–75}$$

where $a = \det[a_{\alpha\beta}]$. By the use of (2-73) and (2-75) the vector products in (2-6) can be written in the following compact forms:

$$\left.\begin{aligned} \mathbf{n}\times\mathbf{a}_\alpha &= \sqrt{a}\,e_{\alpha\beta}\,\mathbf{a}^\beta \\ \mathbf{a}_\alpha\times\mathbf{a}_\beta &= \sqrt{a}\,e_{\alpha\beta}\,\mathbf{n} \end{aligned}\right\} \tag{2–76}$$

Further, the following component identities may be derived:

$$\left.\begin{aligned} a_{\alpha\beta} &= a\,e_{\alpha\gamma}\,a^{\gamma\delta}\,e_{\beta\delta} \\ a^{\alpha\beta} &= \frac{1}{a}\,e^{\gamma\alpha}\,a_{\gamma\delta}\,e^{\delta\beta} \\ e_{\alpha\beta} &= \frac{1}{a}\,a_{\alpha\gamma}\,e^{\delta\gamma}\,a_{\delta\beta} \\ e^{\alpha\beta} &= a\,a^{\alpha\gamma}\,e_{\delta\gamma}\,a^{\delta\beta} \end{aligned}\right\} \tag{2–77}$$

The first relation in (2-77) follows from the matrix identities:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix}^{-1} = a\begin{bmatrix} a^{22} & -a^{12} \\ -a^{21} & a^{11} \end{bmatrix} = a\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T \tag{2–78}$$

where it has been used that $\det[a^{\alpha\beta}] = 1/\det[a_{\alpha\beta}] = \frac{1}{a}$. The proof of the remaining relations in (2-77), which is left an exercise, may be carried out by simple matrix operations similar to (2-78).

Next, a three-dimensional vector field $\mathbf{v} = \mathbf{v}(\theta^1,\theta^2)$ is considered, defined on the parameter domain $(\theta^1,\theta^2) \in \omega$. Hence, a vector is connected to each point of the surface Ω. Since, $v^3 = v_3 \neq 0$, $\mathbf{v}$ is not a surface vector. The following representations prevail:

$$\mathbf{v} = v^\alpha\,\mathbf{a}_\alpha + v^3\,\mathbf{n} = v_\alpha\,\mathbf{a}^\alpha + v_3\,\mathbf{n} \tag{2–79}$$

Since, $\frac{\partial\mathbf{v}}{\partial\theta^3} = \mathbf{0}$, the gradient becomes, cf. (1-56), (1-64), (1-65):

$$\boldsymbol{\nabla}\mathbf{v} = \frac{\partial\mathbf{v}}{\partial\theta^\beta}\,\mathbf{a}^\beta = v^j{}_{;\beta}\,\mathbf{a}_j\mathbf{a}^\beta = v_{j;\beta}\,\mathbf{a}^j\mathbf{a}^\beta \tag{2–80}$$

Similarly, (1-66) becomes:

$$\left.\begin{aligned} \frac{\partial\mathbf{v}}{\partial\theta^\beta} &= v^j{}_{;\beta}\,\mathbf{a}_j = v^\alpha{}_{;\beta}\,\mathbf{a}_\alpha + v^3{}_{;\beta}\,\mathbf{n} \\ \frac{\partial\mathbf{v}}{\partial\theta^\beta} &= v_{j;\beta}\,\mathbf{a}^j = v_{\alpha;\beta}\,\mathbf{a}^\alpha + v_{3;\beta}\,\mathbf{n} \end{aligned}\right\} \tag{2–81}$$

where, cf. (1-65):

$$\left.\begin{aligned} v^{\alpha}{}_{;\beta} &= \frac{\partial v^{\alpha}}{\partial \theta^{\beta}} + \begin{Bmatrix} \alpha \\ \beta \;\; \gamma \end{Bmatrix} v^{\gamma} + \begin{Bmatrix} \alpha \\ \beta \;\; 3 \end{Bmatrix} v^{3} \\ v^{3}{}_{;\beta} &= \frac{\partial v^{3}}{\partial \theta^{\beta}} + \begin{Bmatrix} 3 \\ \beta \;\; \gamma \end{Bmatrix} v^{\gamma} + \begin{Bmatrix} 3 \\ \beta \;\; 3 \end{Bmatrix} v^{3} \\ v_{\alpha;\beta} &= \frac{\partial v_{\alpha}}{\partial \theta^{\beta}} - \begin{Bmatrix} \gamma \\ \alpha \;\; \beta \end{Bmatrix} v_{\gamma} - \begin{Bmatrix} 3 \\ \alpha \;\; \beta \end{Bmatrix} v_{3} \\ v_{3;\beta} &= \frac{\partial v_{3}}{\partial \theta^{\beta}} - \begin{Bmatrix} \gamma \\ 3 \;\; \beta \end{Bmatrix} v_{\gamma} - \begin{Bmatrix} 3 \\ 3 \;\; \beta \end{Bmatrix} v_{3} \end{aligned}\right\} \tag{2–82}$$

The right hand sides of (2-81) both consist of a surface vector and a vector in the normal direction. The surface and normal vectors of these representations must pair-wise be equal corresponding to:

$$\left.\begin{aligned} v^{\alpha}{}_{;\beta}\, \mathbf{a}_{\alpha} &= v_{\alpha;\beta}\, \mathbf{a}^{\alpha} \\ v^{3}{}_{;\beta}\, \mathbf{n} &= v_{3;\beta}\, \mathbf{n} \end{aligned}\right\} \quad \Rightarrow$$

$$\left.\begin{aligned} v^{\alpha}{}_{;\beta} &= a^{\alpha\gamma}\, v_{\gamma;\beta} \\ v_{\alpha;\beta} &= a_{\alpha\gamma}\, v^{\gamma}{}_{;\beta} \\ v_{3;\beta} &= v^{3}{}_{;\beta} \end{aligned}\right\} \tag{2–83}$$

Using $v_3 = v^3$ and $v^3{}_{;\beta} = v_{3;\beta}$ it follows from (2-82) that $\left\{ {3 \atop \beta\;\;3} \right\} = \left\{ {3 \atop 3\;\;\beta} \right\} = -\left\{ {3 \atop 3\;\;\beta} \right\} \Rightarrow \left\{ {3 \atop 3\;\;\beta} \right\} = 0$. Further, $b_{\alpha\beta} = \left\{ {3 \atop \alpha\;\;\beta} \right\}$, cf. (2-46). Then, withdrawal of the second equation in (2-82) from the fourth equation provides the following relations for the components of the curvature tensor:

$$-\begin{Bmatrix} \gamma \\ 3 \;\; \beta \end{Bmatrix} v_{\gamma} = \begin{Bmatrix} 3 \\ \beta \;\; \gamma \end{Bmatrix} v^{\gamma} = b_{\beta\gamma} v^{\gamma} \qquad \Rightarrow$$

$$\left.\begin{aligned} -\begin{Bmatrix} \alpha \\ 3 \;\; \beta \end{Bmatrix} a_{\alpha\gamma}\, v^{\gamma} &= b_{\beta\gamma}\, v^{\gamma} \\ -\begin{Bmatrix} \gamma \\ 3 \;\; \beta \end{Bmatrix} v_{\gamma} &= b_{\beta\alpha}\, a^{\alpha\gamma}\, v_{\gamma} \end{aligned}\right\} \qquad \Rightarrow$$

$$\left.\begin{aligned} b_{\beta\gamma} &= -\begin{Bmatrix} \alpha \\ 3 \;\; \beta \end{Bmatrix} a_{\alpha\gamma} \\ b_{\beta}{}^{\gamma} &= -\begin{Bmatrix} \gamma \\ 3 \;\; \beta \end{Bmatrix} \end{aligned}\right\} \tag{2–84}$$

2.3 Codazzi's equations

From (2-67), (2-71) and (2-72) follows:

$$\frac{\partial b_{\alpha\beta}}{\partial\theta^\gamma} = -\frac{\partial}{\partial\theta^\gamma}\left(\mathbf{a}_\alpha\cdot\frac{\partial\mathbf{n}}{\partial\theta^\beta}\right) = -\frac{\partial\mathbf{a}_\alpha}{\partial\theta^\gamma}\cdot\frac{\partial\mathbf{n}}{\partial\theta^\beta} - \mathbf{a}_\alpha\cdot\frac{\partial^2\mathbf{n}}{\partial\theta^\gamma\partial\theta^\beta} =$$
$$\left(\begin{Bmatrix}\delta\\ \alpha\ \ \gamma\end{Bmatrix}\mathbf{a}_\delta + b_{\alpha\gamma}\,\mathbf{n}\right)\cdot b_{\beta\varepsilon}\,\mathbf{a}^\varepsilon - \mathbf{a}_\alpha\cdot\frac{\partial^2\mathbf{n}}{\partial\theta^\gamma\partial\theta^\beta} = b_{\beta\delta}\begin{Bmatrix}\delta\\ \alpha\ \ \gamma\end{Bmatrix} - \mathbf{a}_\alpha\cdot\frac{\partial^2\mathbf{n}}{\partial\theta^\gamma\partial\theta^\beta} \tag{2–85}$$

Interchange of the indices β and γ in (2-85) provides:

$$\frac{\partial b_{\alpha\gamma}}{\partial\theta^\beta} = b_{\gamma\delta}\begin{Bmatrix}\delta\\ \alpha\ \ \beta\end{Bmatrix} - \mathbf{a}_\alpha\cdot\frac{\partial^2\mathbf{n}}{\partial\theta^\beta\partial\theta^\gamma} \tag{2–86}$$

Withdrawal of (2-86) from (2-85) gives the relation:

$$\frac{\partial b_{\alpha\beta}}{\partial\theta^\gamma} - b_{\beta\delta}\begin{Bmatrix}\delta\\ \alpha\ \ \gamma\end{Bmatrix} = \frac{\partial b_{\alpha\gamma}}{\partial\theta^\beta} - b_{\gamma\delta}\begin{Bmatrix}\delta\\ \alpha\ \ \beta\end{Bmatrix} \tag{2–87}$$

(2-87) is trivially fulfilled for $\beta=\gamma$. Further, the index combinations $(\alpha,\beta,\gamma)=(1,1,2)$ and $(\alpha,\beta,\gamma)=(1,2,1)$, as well as the index combinations $(\alpha,\beta,\gamma)=(2,1,2)$ and $(\alpha,\beta,\gamma)=(2,2,1)$, both lead to the same relation. Hence, merely two non-trivial and independent relations exist, which are taken corresponding to the index combinations $(\alpha,\beta,\gamma)=(1,1,2)$ and $(\alpha,\beta,\gamma)=(2,2,1)$, leading to the following relations:

$$\left.\begin{aligned}\frac{\partial b_{11}}{\partial\theta^2} - b_{1\delta}\begin{Bmatrix}\delta\\ 1\ \ 2\end{Bmatrix} &= \frac{\partial b_{12}}{\partial\theta^1} - b_{2\delta}\begin{Bmatrix}\delta\\ 1\ \ 1\end{Bmatrix}\\ \frac{\partial b_{22}}{\partial\theta^1} - b_{2\delta}\begin{Bmatrix}\delta\\ 2\ \ 1\end{Bmatrix} &= \frac{\partial b_{21}}{\partial\theta^2} - b_{1\delta}\begin{Bmatrix}\delta\\ 2\ \ 2\end{Bmatrix}\end{aligned}\right\} \tag{2–88}$$

The relations (2-88) are called *Codazzi's equations*. It can be shown that arbitrary component functions $a_{\alpha\beta}=a_{\alpha\beta}(\theta^1,\theta^2)$ and $b_{\alpha\beta}=b_{\alpha\beta}(\theta^1,\theta^2)$ of the curvilinear parameters θ^1 and θ^2 uniquely determine a surface (saved an arbitrary rigid body translation and rotation), which has $a_{\alpha\beta}\,d\theta^\alpha d\theta^\beta$ and $b_{\alpha\beta}\,d\theta^\alpha d\theta^\beta$ as its first and second fundamental form, if and only if these components fulfill the Codazzi's equations (2-88) and the Gauss's equation (2-52), (Spain, 1965).

In a principal curvature coordinate system we have $a_{12}=a_{21}=0$ and $b_{12}=b_{21}=0$. In this case the Codazzi's equations reduce to:

$$\left.\begin{aligned}\frac{\partial}{\partial\theta^2}\left(\frac{a_1}{R_1}\right) &= \frac{1}{R_2}\frac{\partial a_1}{\partial\theta^2}\\ \frac{\partial}{\partial\theta^1}\left(\frac{a_2}{R_2}\right) &= \frac{1}{R_1}\frac{\partial a_2}{\partial\theta^1}\end{aligned}\right\} \tag{2–89}$$

where a_1 and a_2 are given by (2-74), and R_1 and R_2 are the principal curvature radii . A proof of (2-89) has been given in Box 1.2.

Box 2.1: Proof of (2-89)

Since $b_{12} = b_{21} = 0$ in the principal curvature coordinate system, (2-88) can be reduced to:

$$\left.\begin{aligned} \frac{\partial b_{11}}{\partial \theta^2} &= b_{11} \begin{Bmatrix} 1 \\ 1 \;\; 2 \end{Bmatrix} - b_{22} \begin{Bmatrix} 2 \\ 1 \;\; 1 \end{Bmatrix} \\ \frac{\partial b_{22}}{\partial \theta^1} &= b_{22} \begin{Bmatrix} 2 \\ 2 \;\; 1 \end{Bmatrix} - b_{11} \begin{Bmatrix} 1 \\ 2 \;\; 2 \end{Bmatrix} \end{aligned}\right\} \tag{2–90}$$

Next, the Christoffel symbols in (2-90) are expressed in terms of the covariant and contravariant components of the identity tensor as given by the defining equation (1-58). Due to the principal curvature coordinates these components become $a_{\alpha\beta} = a^{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $a^{(\alpha)(\alpha)} = \frac{1}{a_{(\alpha)(\alpha)}}$. Then:

$$\left.\begin{aligned} \begin{Bmatrix} 1 \\ 1 \;\; 2 \end{Bmatrix} &= \frac{1}{2} a^{11} \left(\frac{\partial a_{11}}{\partial \theta^2} + 0 - 0 \right) = \frac{1}{2} \frac{1}{a_{11}} \frac{\partial a_{11}}{\partial \theta^2} \\ \begin{Bmatrix} 2 \\ 1 \;\; 1 \end{Bmatrix} &= \frac{1}{2} a^{22} \left(0 + 0 - \frac{\partial a_{11}}{\partial \theta^2} \right) = -\frac{1}{2} \frac{1}{a_{22}} \frac{\partial a_{11}}{\partial \theta^2} \\ \begin{Bmatrix} 2 \\ 2 \;\; 1 \end{Bmatrix} &= \frac{1}{2} a^{22} \left(\frac{\partial a_{22}}{\partial \theta^1} + 0 - 0 \right) = \frac{1}{2} \frac{1}{a_{22}} \frac{\partial a_{22}}{\partial \theta^1} \\ \begin{Bmatrix} 1 \\ 2 \;\; 2 \end{Bmatrix} &= \frac{1}{2} a^{11} \left(0 + 0 - \frac{\partial a_{22}}{\partial \theta^1} \right) = -\frac{1}{2} \frac{1}{a_{11}} \frac{\partial a_{22}}{\partial \theta^1} \end{aligned}\right\} \tag{2–91}$$

Further, from (2-66) follows that $b_{11} = (1/R_1)a_{11}$ and $b_{22} = (1/R_2)a_{22}$. Then, insertion of (2-91) in (2-90) provides:

$$\left.\begin{aligned} \frac{\partial}{\partial \theta^2} \left(\frac{a_{11}}{R_1} \right) &= \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{\partial a_{11}}{\partial \theta^2} \\ \frac{\partial}{\partial \theta^1} \left(\frac{a_{22}}{R_2} \right) &= \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{\partial a_{22}}{\partial \theta^1} \end{aligned}\right\} \tag{2–92}$$

Introduction of $(a_1)^2 = a_{11}$ and $(a_2)^2 = a_{22}$ provides the identities:

$$\left.\begin{aligned}\frac{\partial}{\partial\theta^2}\left(\frac{a_{11}}{R_1}\right) &= \frac{\partial}{\partial\theta^2}\left(a_1\frac{a_1}{R_1}\right) = \frac{a_1}{R_1}\frac{\partial a_1}{\partial\theta^2} + a_1\frac{\partial}{\partial\theta^2}\left(\frac{a_1}{R_1}\right)\\ \frac{\partial}{\partial\theta^1}\left(\frac{a_{22}}{R_2}\right) &= \frac{\partial}{\partial\theta^1}\left(a_2\frac{a_2}{R_2}\right) = \frac{a_2}{R_2}\frac{\partial a_2}{\partial\theta^1} + a_2\frac{\partial}{\partial\theta^1}\left(\frac{a_2}{R_2}\right)\\ \frac{\partial a_{11}}{\partial\theta^2} &= 2a_1\frac{\partial a_1}{\partial\theta^2}\\ \frac{\partial a_{22}}{\partial\theta^1} &= 2a_2\frac{\partial a_2}{\partial\theta^1}\end{aligned}\right\} \qquad (2–93)$$

Finally, (2-89) is obtained by insertion of the results in (2-93) into (2-92).

Example 2.1: Identity tensor, curvature tensor, Riemann-Christoffel tensor and principal curvatures on a spherical surface

A spherical coordinate system is used, and let $r = \theta^3$ denote the radius of the sphere. Then it follows from (1-85) that the contravariant components of the surface identity tensor becomes:

$$[a_{\alpha\beta}] = \begin{bmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta^1 \end{bmatrix} \qquad (2–94)$$

The contravariant components of the curvature tensor follow from (1-86) and (2-46):

$$[b_{\alpha\beta}] = \begin{bmatrix} -r & 0 \\ 0 & -r\sin^2\theta^1 \end{bmatrix} \qquad (2–95)$$

Since both $[a_{\alpha\beta}$ and $[b_{\alpha\beta}]$ are diagonal, it is concluded that the spherical coordinate system is a principal curvature coordinate system. The principal curvatures and related principal curvature radii follow from (2-40), (2-66):

$$\begin{aligned}\kappa_1 &= \kappa_2 = -\frac{1}{r} \quad \Rightarrow\\ R_1 &= R_2 = -r\end{aligned} \qquad (2–96)$$

κ_1 and κ_2 are negative, because unit surface normal vector is directed away from the curvature center, which in this case is identical to the origin of the Cartesian coordinate system.

The selected independent component of the Riemann-Christoffel tensor becomes, cf. (2-52):

$$B_{1212} = r^2\sin^2\theta^1 \qquad (2–97)$$

Using the calculated Christoffel symbols in Eq. (1-86) it follows that the left and right hand sides of the first Codazzi equation in (2-90) vanish. The second Codazzi equation provides the result $-2r\cos(\theta^1)\sin(\theta^1)$ on the left and right hand sides.

2.4 Surface area elements

In structural analysis of shells it may be of interest to know the relation corresponding element areas on the upper and lower surface. This problem is studied in this section.

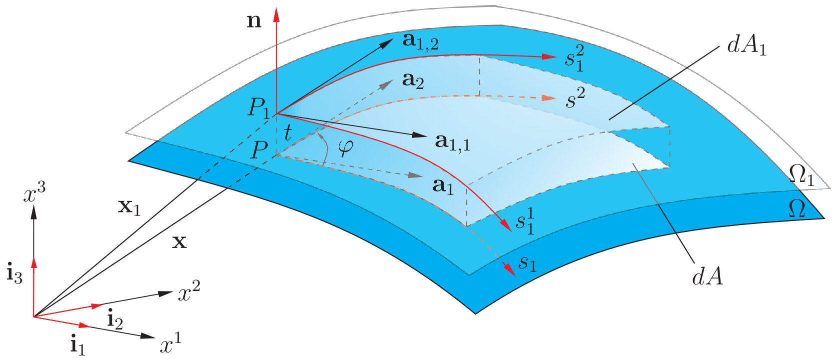

Fig. 2–5 Area elements in parallel surfaces.

Fig. 2-5 shows two surfaces Ω and Ω_1, described by the same parameters θ^1 and θ^2. The points P and P_1 denotes the mapping points for the same parameters, specified by the position vectors $\mathbf{x} = \mathbf{x}(\theta^1, \theta^2)$ and $\mathbf{x}_1 = \mathbf{x}_1(\theta^1, \theta^2)$, respectively. Further, the unit surface normal vectors related to the two surfaces at the said points are assumed to be identical and equal to $\mathbf{n}$. Further, it is assumed that the distance between any such corresponding points P and P_1 on the surfaces are equal to t. On the surfaces arc length coordinate coordinate systems (s^1, s^2) and (s_1^1, s_1^2) are introduced with origin at P and P_1. The related surface covariant base vectors are denoted $\mathbf{a}_\alpha$ and $\mathbf{a}_{1,\alpha}$, respectively. Then, the position vectors are related as follows:

$$\mathbf{x}_1 = \mathbf{x} + t\,\mathbf{n} \tag{2–98}$$

dA denotes a differential area element of shape as a parallelogram with sides of the length ds^1 and ds^2 parallel to $\mathbf{a}_1$ and $\mathbf{a}_2$, see Fig. 2-5. The length of the sides follows from (2-19):

$$\left.\begin{aligned} ds^1 &= \sqrt{a_{11}}\, d\theta^1 \\ ds^2 &= \sqrt{a_{22}}\, d\theta^2 \end{aligned}\right\} \tag{2–99}$$

Let φ denote the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$. Then, cf. (2-5), (2-74):

$$dA = ds^1 ds^2 \sin\varphi = \sqrt{a_{11}}\sqrt{a_{22}}\,\sin\varphi\, d\theta^1 d\theta^2 = \sqrt{a_{11}a_{22} - (a_{12})^2}\, d\theta^1 d\theta^2 = \sqrt{a}\, d\theta^1 d\theta^2 \tag{2–100}$$

The surface covariant base vectors $\mathbf{a}_{1,\alpha}$ follows from (2-3), (2-69), (2-70):

$$\mathbf{a}_{1,\alpha} = \frac{\partial \mathbf{x}_1}{\partial \theta^\alpha} = \frac{\partial \mathbf{x}}{\partial \theta^\alpha} + t\,\frac{\partial \mathbf{a}_3}{\partial \theta^\alpha} = \mathbf{a}_\alpha - t\, b^\delta{}_\alpha\, \mathbf{a}_\delta = \left(\delta_\alpha^\delta - t\, b^\delta{}_\alpha\right) \mathbf{a}_\delta \tag{2–101}$$

Then, the contravariant components $a_{1,\alpha\beta}$ of the surface identity tensor $\mathbf{a}_1$ of Ω_1 becomes:

$$a_{1,\alpha\beta} \;=\; \mathbf{a}_{1,\alpha} \cdot \mathbf{a}_{1,\beta} \;=\; \left(\delta^\delta_\alpha - t\, b^\delta{}_\alpha\right) \left(\delta^\gamma_\beta - t\, b^\gamma{}_\beta\right) \mathbf{a}_\delta \cdot \mathbf{a}_\gamma \;=\; \left(\delta^\delta_\alpha - t\, b^\delta{}_\alpha\right) \left(\delta^\gamma_\beta - t\, b^\gamma{}_\beta\right) a_{\delta\gamma} =$$
$$a^{\gamma\delta}\,(a_{\gamma\alpha} - t\, b_{\gamma\alpha})\,(a_{\delta\beta} - t\, b_{\delta\beta}) \;=\; a^{\gamma\delta}\,(a_{\alpha\gamma} - t\, b_{\alpha\gamma})\,(a_{\beta\delta} - t\, b_{\beta\delta}) \tag{2–102}$$

The differential area element dA_1 on Ω_1, corresponding to dA on Ω is given as:

$$dA_1 \;=\; \sqrt{a_1}\, d\theta^1 d\theta^2 \tag{2–103}$$

where $a_1 = \det[a_{1,\alpha\beta}]$ follows from (2-102):

$$a_1 \;=\; \det\left[a^{\gamma\delta}\right] \det\left[a_{\alpha\gamma} - t\, b_{\alpha\gamma}\right] \det\left[a_{\beta\delta} - t\, b_{\beta\delta}\right] \;=\; \frac{1}{a}\Big(\det\left[a_{\alpha\beta} - t\, b_{\alpha\beta}\right]\Big)^2 \;=$$
$$\frac{1}{a}\Big(\det\left[a_{\alpha\gamma}\right] \det\left[\delta^\gamma_\beta - t\, b^\gamma{}_\beta\right]\Big)^2 \;=\; \frac{\big(\det\left[a_{\alpha\gamma}\right]\big)^2}{a}\Big(\det\left[\delta^\gamma_\beta - t\, b^\gamma{}_\beta\right]\Big)^2 \;=$$
$$a\left(1 - b^\alpha{}_\alpha t + \frac{b}{a}\, t^2\right)^2 \;=\; a\left(1 - 2H\, t + K\, t^2\right)^2 \tag{2–104}$$

where (2-60) and (2-61) have been used. Further, the mean curvature H and the Gaussian curvature K as given by (2-63) and (2-64) have been introduced. Then the ratio between the differential area elements becomes:

$$\frac{dA_1}{dA} \;=\; \frac{\sqrt{a_1}}{\sqrt{a}} \;=\; 1 - 2H\, t + K\, t^2 \tag{2–105}$$

2.5 Exercises

2.1 Prove that the surface Riemann tensor ${B^\delta}_{\alpha\beta\gamma}$ merely contains one independent component.

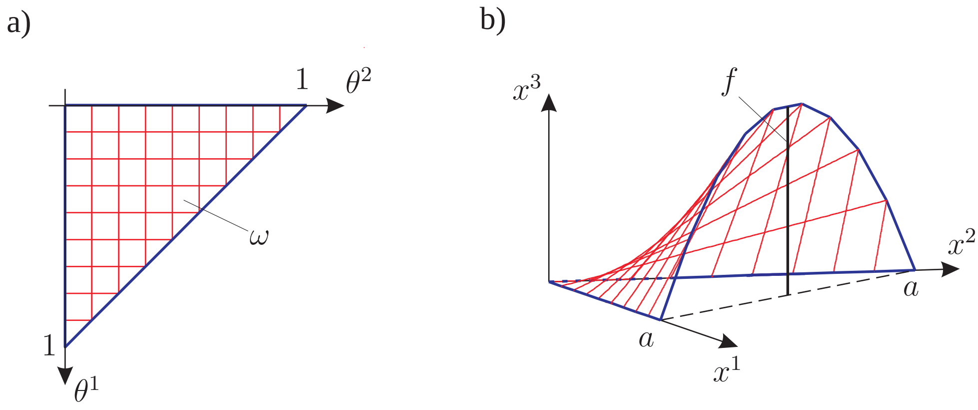

Fig. 2–6 a) Parameter domain. b) Shell surface.

2.2 The mid-surface of a shell structure is defined by the parametric description:

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} a\,\theta^1 \\ a\,\theta^2 \\ 4f\,\theta^1\theta^2 \end{bmatrix}$$

where the parameter domain ω has been indicated on Fig. 2-6a. a and f are parameters defining the length and the height of the shell structure as shown on Fig. 2-6b.

(a.) Calculate the covariant and contravariant base vectors $\mathbf{g}_j$ and $\mathbf{g}^j$ in terms of the Cartesian base vectors.

(b.) Calculate the contravariant components of the surface identity tensor and the curvature tensor, and determine principal curvature, the mean curvature, the Gaussian curvature and the principal curvature directions.

(c.) Formulate Codazzi's equations for the surface in terms of the selected curvilinear coordinates.

2.3 Consider a shell structure with the thickness t, and let Ω_0 denote the so-called *middle surface*, and Ω_1 and Ω_{-1} the upper and lower surfaces.

(a.) Calculate the fraction $\frac{dA^1}{dA_{-1}}$ of corresponding differential area elements dA_1 and dA_{-1} on the upper and lower surfaces expressed by the mean curvature H and the Gaussian curvature K of the middle surface.

CHAPTER 3
Dynamics

3.1 Equation of motion of a mass particle

A particle with the mass m is moving in the three-dimensional Euclidian space under the influence of a time-dependent force vector $\mathbf{f}(t)$. The position along the trajectory of the particle is given by the position vector $\mathbf{x}(t)$ with covariant curvilinear coordinates θ^j.

The increment of the position vector $d\mathbf{x}(t) = \mathbf{x}(t+dt) - \mathbf{x}(t)$ during the time interval $]t, t+dt]$ is given as, cf. (1-8):

$$d\mathbf{x}(t) = \frac{\partial \mathbf{x}(t)}{\partial \theta^j} d\theta^j(t) = \mathbf{g}_j\big(\theta^l(t)\big)\, d\theta^j(t) \tag{3–1}$$

The *velocity vector* of the particle is given as $\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}$. The coordinate expansion in the curvilinear vector base follows from (3-1):

$$\mathbf{v}(t) = \mathbf{g}_j\big(\boldsymbol{\theta}(t)\big)\frac{d\theta^j(t)}{dt} \tag{3–2}$$

Hence, the covariant components of the velocity vector becomes $v^j(t) = \frac{dx^j(t)}{dt}$.

The *acceleration vector* is given as $\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}$. Differentiation of (3-2) provides the following representation of the acceleration vector in the covariant vector basis:

$$\mathbf{a}(t) = \mathbf{g}_j\big(\boldsymbol{\theta}(t)\big)\frac{d^2\theta^j}{dt^2} + \frac{\partial \mathbf{g}_l\big(\boldsymbol{\theta}(t)\big)}{\partial \theta^k}\frac{d\theta^k}{dt}\frac{dx\theta l}{dt} =$$
$$\left(\frac{d^2\theta^j}{dt^2} + \begin{Bmatrix} j \\ k \ \ l \end{Bmatrix}\frac{d\theta^k}{dt}\frac{d\theta^l}{dt}\right)\mathbf{g}_j\big(\boldsymbol{\theta}(t)\big) \tag{3–3}$$

In (3-3) the dummy index j has been changed to l in the last part of the second statement. Further, the relation $\frac{\partial \mathbf{g}_l}{\partial \theta^k} = \left\{\begin{smallmatrix} j \\ k \ \ l \end{smallmatrix}\right\}\mathbf{g}_j$ has been used in the last statement, cf. (1-57).

The equation of motion is given by *Newton's 2nd law of motion*:

$$m\,\mathbf{a}(t) = \mathbf{f}(t) \tag{3–4}$$

The components in the curvilinear covariant basis become, cf. (3-3):

$$
\begin{aligned}
& m\left(\frac{d^2\theta^k}{dt^2} + \begin{Bmatrix} j \\ k \;\; l \end{Bmatrix} \frac{d\theta^k}{dt}\frac{d\theta^l}{dt}\right)\mathbf{g}_j \;=\; f^j(t)\,\mathbf{g}_j \qquad\qquad \Rightarrow \\
& m\left(\frac{d^2\theta^k}{dt^2} + \begin{Bmatrix} j \\ k \;\; l \end{Bmatrix} \frac{d\theta^k}{dt}\frac{d\theta^l}{dt}\right) \;=\; f^j(t)
\end{aligned}
\tag{3–5}
$$

Assume that the particle is moving without any external load, i.e. $\mathbf{f}(t) \equiv \mathbf{0}$. Then, $\mathbf{a}(t) \equiv \mathbf{0}$, and the velocity vector becomes $\mathbf{v}(t) = \mathbf{v}_0$. The position vector $\mathbf{x}(t)$ becomes:

$$
\mathbf{x}(t) \;=\; \mathbf{x}_0 \;+\; \mathbf{v}_0\,(t - t_0) \tag{3–6}
$$

where $\mathbf{x}_0 = \mathbf{x}(t_0)$. (3-6) indicates the parametrization of s straight line, which is a geodesic in the three dimensional Euclidian space. Hence, the particle moves along a geodesic, whenever $\mathbf{a} \equiv \mathbf{0}$. The constant speed of the particle is given by $v_0 \;=\; \sqrt{\mathbf{v}_0 \cdot \mathbf{v}_0}$.

Next, assume that the motion of the particle is constrained to take place on a surface. Further the curvilinear coordinate θ^3 is assumed to be in the normal direction. Then, the position on the surface is given by the position vector $\mathbf{x}(\theta_1, \theta_2)$, cf. (2-2). The surface covariant vector base $\mathbf{a}_\alpha(\theta_1, \theta_2)$ is defined by (2-3), and the unit normal vector $\mathbf{n}(\theta_1, \theta_2)$ is given by (2-4).

Due to the constrain to the surface the motion of the particle in the normal direction is zero, i.e. $\theta^3 \equiv 0$. Then, (3-5) becomes:

$$
\left.
\begin{aligned}
& m\left(\frac{d^2\theta^\alpha}{dt^2} + \begin{Bmatrix} \alpha \\ \beta \;\; \gamma \end{Bmatrix} \frac{d\theta^\beta}{dt}\frac{d\theta^\gamma}{dt}\right) \;=\; f^\alpha(t) \quad , \quad \alpha = 1, 2 \\
& m \begin{Bmatrix} 3 \\ \alpha \;\; \beta \end{Bmatrix} \frac{d\theta^\alpha}{dt}\frac{d\theta^\beta}{dt} \qquad\qquad\quad \;=\; f^3(t) \quad , \quad j = 3
\end{aligned}
\right\}
\tag{3–7}
$$

In the first equation (3-7) it has been used that $\frac{d\theta^3}{dt} \equiv 0$.

$f^3(t)$ in the second equation (3-8) may be written as $f^3(t) \;=\; f_r^3(t) + f_e^3(t)$ where $f_r^3(t)$ and $f_e^3(t)$ indicate the covariant components of the reaction force vector from the surface and the external force vector on the particle in the direction of the unit normal vector, respectively. Using (2-46) the following relation is obtained for $f_r^3(t)$:

$$
f_r^3(t) \;=\; m\, b_{\alpha\beta}\,\frac{d\theta^\alpha}{dt}\frac{d\theta^\beta}{dt} \;-\; f_e^3(t) \tag{3–8}
$$

The first term on the right-hand side of (3-8) indicates the centrifugal load due to the curvature of the surface.

If the surface is smooth, $f_r^3(t)$ will not induce any friction force against the motion of the particle. However for non-smooth surfaces, a friction force $\mu f_r^3(t)$ is acting in the negative direction of the velocity vector $\mathbf{v}(t)$ of the particle, where μ indicates a friction coefficient. Then, $f^\alpha(t)$ is given as:

$$f^\alpha(t) = f_e^\alpha(t) - \mu f_r^3(t) \frac{v^\alpha(t)}{|\mathbf{v}(t)|} \tag{3–9}$$

where $f_e^\alpha(t)$ and $v^\alpha(t) = \frac{d\theta^\alpha(t)}{dt}$ signify the covariant components of the external force vector and the velocity vector in the surface covariant base.

3.2 Nonlinear multi-degree-of-freedom systems

The motion of a *multi-degree-of-freedom system* is described by an N-dimensional displacement vector $\boldsymbol{\theta}(t)$. In mechanics, the covariant curvilinear coordinates θ^l are referred to as the *degrees of freedom* or the *generalized displacements* of the system. These are typically made up of nodal displacements and rotations of a finite element model of a continuous media. The set of all possible positions with due consideration to possible kinematic and mathematics constraints form a subset of the N-dimensional Riemann space known as the *configuration space*.

The *kinetic energy* of a discrete dynamic system is given as:

$$T(t) = \frac{1}{2}\dot{\boldsymbol{\theta}}(t) \cdot \mathbf{M}\big(\boldsymbol{\theta}(t)\big) \cdot \dot{\boldsymbol{\theta}}(t) \tag{3–10}$$

where $\dot{\boldsymbol{\theta}}(t) = \frac{d}{dt}\boldsymbol{\theta}(t)$ is the velocity vector, and the symmetric tensor $\mathbf{M}\big(\boldsymbol{\theta}(t)\big) = \mathbf{M}^T\big(\boldsymbol{\theta}(t)\big)$ is known as the *mass tensor*. The mass tensor is independent of $\boldsymbol{\theta}(t)$ for linear systems.

The contravariant components of the mass tensor $\mathbf{M}\big(\boldsymbol{\theta}(t)\big)$ are denoted as M_{jk}, and the covariant components of the inverse mass tensor $\mathbf{M}^{-1}\big(\boldsymbol{\theta}(t)\big)$ are denoted as $M^{-1,\,jk}$. Hence, $M^{-1,\,kl}\,M_{lj} = \delta_j^k$.

The load vector $\mathbf{f}(t)$ work conjugated to $\boldsymbol{\theta}(t)$ may be decomposed into a *conservative force vector* $\mathbf{f}_\mathrm{c}(t) = \mathbf{f}_\mathrm{c}\big(\boldsymbol{\theta}(t)\big)$ and a *non-conservative force vector* $\mathbf{f}_\mathrm{nc}(t) = \mathbf{f}_\mathrm{nc}\big(t, \boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)\big)$, i.e.:

$$\mathbf{f}(t) = \mathbf{f}_\mathrm{c}(t) + \mathbf{f}_\mathrm{nc}(t) \tag{3–11}$$

The contravariant components $f_{\mathrm{c},m}(t)$ of the conservative load vector are determined as the negative gradient of a related *potential energy function* $V\big(\boldsymbol{\theta}(t)\big)$:

$$f_{\mathrm{c},m}(t) = -\frac{\partial V\big(\boldsymbol{\theta}(t)\big)}{\partial \theta^m} \tag{3–12}$$

As seen the gradient is taken with respect to the covariant displacement coordinate $\theta^m(t)$, which is the work conjugated degree of freedom to contravariant load component $f_{\mathrm{c},m}(t)$.

The *Lagrangian* $L\big(\boldsymbol{\theta}(t),\, \dot{\boldsymbol{\theta}}(t)\big)$ of the system is defined as:

$$L\big(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)\big) = T\big(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)\big) - V\big(\boldsymbol{\theta}(t)\big) \tag{3–13}$$

Then, the equations of motion in contravariant coordinates is given by the *Euler-Lagrange stationarity condition* of the so-called action integral, (Pars, 1964):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}^m}\right) - \frac{\partial L}{\partial \theta^m} = f_{\mathrm{nc},m}(t) \quad , \quad m = 1, \ldots, N \tag{3–14}$$

By the use of (3-10), (3-12), (3-13), Eq. (3-14) may be written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}^m}\right) - \frac{\partial L}{\partial \theta^m} = M_{mk}\frac{d^2\theta^k}{dt^2} + \frac{\partial M_{mk}}{\partial \theta^l}\frac{d\theta^k}{dt}\frac{d\theta^l}{dt} - \frac{1}{2}\frac{\partial M_{kl}}{\partial \theta^k}\frac{d\theta^k}{dt}\frac{d\theta^l}{dt} + \frac{\partial V}{\partial \theta^m} =$$
$$M_{mk}\frac{d^2\theta^k}{dt^2} + \frac{1}{2}\left(\frac{\partial M_{mk}}{\partial \theta^l} + \frac{\partial M_{ml}}{\partial \theta^k} - \frac{\partial M_{kl}}{\partial \theta^m}\right)\frac{d\theta^k}{dt}\frac{d\theta^l}{dt} - f_{\mathrm{c},m}(t) = f_{\mathrm{nc},m}(t) \tag{3–15}$$

where the symmetry property $M_{jk} = M_{kj}$ has been used.

By pre-multiplication with $M^{-1,\,jm}$ and contraction over the index m, (3-15) may be written on the compact form:

$$\frac{d^2\theta^j}{dt^2} + \left\{ \begin{matrix} j \\ k \;\; l \end{matrix} \right\}_M \frac{d\theta^k}{dt}\frac{d\theta^l}{dt} = M^{-1,\,jm}\Big(f_{\mathrm{c},m}(t) + f_{\mathrm{nc},m}(t)\Big) \quad , \quad j = 1, \ldots, N \tag{3–16}$$

where the symbol $\left\{ \begin{smallmatrix} j \\ k \;\; l \end{smallmatrix} \right\}_M$ is defined as:

$$\left\{ \begin{matrix} j \\ k \;\; l \end{matrix} \right\}_M = \frac{1}{2}M^{-1,\,jm}\left(\frac{\partial M_{mk}}{\partial \theta^l} + \frac{\partial M_{ml}}{\partial \theta^k} - \frac{\partial M_{kl}}{\partial \theta^m}\right) \tag{3–17}$$

As follows of (1-58), $\left\{ \begin{smallmatrix} j \\ k \;\; l \end{smallmatrix} \right\}_M$ indicates the Christoffel symbol, when the contravariant components g_{kl} of the fundamental tensor $\mathbf{g}$ is replaced by the contravariant components M_{kl} of the mass tensor, and the covariant components g^{jm} is replaced with the covariant components $M^{-1,\,jm}$ of the inverse mass tensor. This does not make $\mathbf{M}$ a fundamental tensor for the present problem. Actually, $\mathbf{g} \cdot \dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}$, whereas $\mathbf{M} \cdot \dot{\boldsymbol{\theta}} \neq \dot{\boldsymbol{\theta}}$. Further, $M^{-1,\,jm}$ is different from the covariant components $M^{jm} = g^{jk}g^{ml}M_{kl}$ of the mass tensor. The specific definition of the Christoffel symbol has been marked by a subscript M.

The terms involving $\left\{ \begin{smallmatrix} j \\ k \;\; l \end{smallmatrix} \right\}_M$ in (3-16) represents gyroscopic centripetal and Coriolis loads on the system.

Example 3.1: Non-linear two-degree-of-freedom mathemathical pendulum with moving support

Fig. 3.1 shows a mass particle m_1 moving on a horizontal smooth plane. The motion from the statical equilibrium position is described by the displacement degree of freedom $\theta^1(t)$. The mass particle is supported by a nonlinear spring with the spring stiffness $k(\theta^1)$ and a nonlinear viscous damper with the damping coefficient $c(\theta^1)$, and is exposed to an external force $F_1(t)$ acting co-directional to $\theta^1(t)$.

The other end of the point mass is connected via a frictionless hinge to a mathematical pendulum consisting of a rigid massless bar of the length l with a point mass m_2 attached at the free end. The position of the pendulum is described by the anti-clockwise rotation $\theta^2(t)$ measured from a vertical line positioned at the displaced point mass

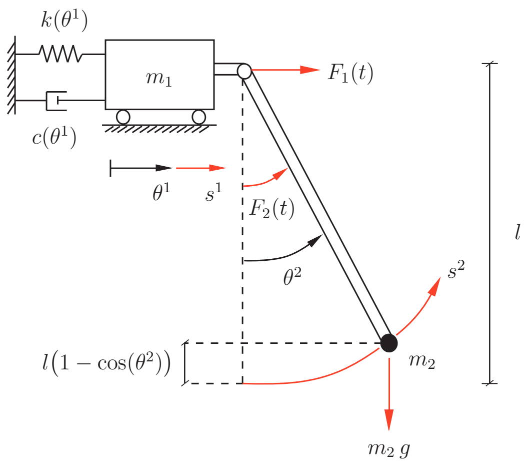

Fig. 3–1 Mathemathical pendulum with moving support.

m_1. The pendulum is exposed to an external moment $F_2(t)$ acting co-directional to $\theta^2(t)$, and to the gravity force $m_2 g$, where g signifies the acceleration of gravity.

The spring stiffness and the damping coefficient are given as:

$$k(\theta^l) \;=\; k_0\left(1+\varepsilon\,(\theta^1)^2\right) \tag{3–18}$$

$$c(\theta^l) \;=\; c_0\left(-1+\left(\frac{\theta^1}{\theta_0}\right)^2\right) \tag{3–19}$$

(3-18) indicates the spring stiffness of a *Duffing oscillator*, and (3-19) is the damping coefficient of a *van der Pol oscillator*. k_0, ε, c_0 and θ_0 are given constants. ε defines the strength of the non-linearity of the spring stiffness, and θ_0 determines the magnitude of the stationary limit cycle of the van der Pol oscillator.

The kinetic energy becomes, see Fig. 3-1:

$$\begin{aligned}&T(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) \;=\; \frac{1}{2}\,m_1\,(\dot\theta^1)^2 \;+\; \frac{1}{2}\,m_2\left(\left(\dot\theta^1+l\,\dot\theta^2\cos(\theta^2)\right)^2 \;+\; \left(l\,\dot\theta^2\sin(\theta^2)\right)^2\right) \;=\\ &\frac{1}{2}\,(m_1+m_2)\,(\dot\theta^1)^2 \;+\; m_2\,l\,\cos(\theta^2)\,\dot\theta^1\dot\theta^2 \;+\; \frac{1}{2}\,m_2\,(l\,\dot\theta^2)^2\end{aligned} \tag{3–20}$$

The potential energy becomes:

$$\begin{aligned}&V(\boldsymbol{\theta}) \;=\; \int_0^{\theta^1} k(u)u\,du \;+\; m_2\,g\,l\left(1-\cos(\theta^2)\right) \;=\\ &\frac{1}{2}\,k_0\left((\theta^1)^2+\frac{1}{2}\,\varepsilon\,(\theta^1)^4\right) \;+\; m_2\,g\,l\left(1-\cos(\theta^2)\right)\end{aligned} \tag{3–21}$$

The contravariant components of the non-conservative force vector becomes:

$$f_{\mathrm{nc},j}(t) \;=\; \begin{cases} F_1(t) \;-\; c_0\left(-1+\left(\dfrac{\theta^1}{\theta_0}\right)^2\right)\dot\theta^1 & ,\quad j=1\\[2ex] F_2(t) & ,\quad j=2\end{cases} \tag{3–22}$$

Then, (3-15) attains the form:

$$\begin{aligned}&\begin{bmatrix} m_1+m_2 & m_2\,l\cos(\theta^2)\\ m_2\,l\cos(\theta^2) & m_2\,l^2\end{bmatrix}\begin{bmatrix}\ddot\theta^1\\ \ddot\theta^2\end{bmatrix} - \begin{bmatrix} m_2\,l\,(\dot\theta^2)^2\sin(\theta^2)\\ m_2\,l\,\dot\theta^1\,\dot\theta^2\,\sin(\theta^2)\end{bmatrix} +\\ &\begin{bmatrix} c_0\left(-1+\left(\frac{\theta^1}{\theta_0}\right)^2\right)\dot\theta^1 \;+\; k_0\left(1+\varepsilon\,(\theta^1)^2\right)\theta^1\\ m_2\,g\,l\,\sin(\theta^2)\end{bmatrix} = \begin{bmatrix}F_1(t)\\ F_2(t)\end{bmatrix}\end{aligned} \tag{3–23}$$

As follows from (3-23), the contravariant components $M_{jk}(\theta^2)$ of $\mathbf{M}(\theta^2)$ and the covariant components $M^{-1,\,jk}(\theta^2)$ of $\mathbf{M}^{-1}(\theta^2)$ are given as:

$$\big[M_{jk}(\theta^2)\big] \;=\; \begin{bmatrix} m_1 + m_2 & m_2\, l \cos\big(\theta^2\big) \\ m_2\, l \cos\big(\theta^2\big) & m_2\, l^2 \end{bmatrix} \tag{3–24}$$

$$\big[M^{-1,\,jk}(\theta^2)\big] \;=\; \frac{1}{\big(m_1 \,+\, m_2\, \sin^2(\theta^2)\big)\, m_2\, l^2} \begin{bmatrix} m_2\, l^2 & -\, m_2\, l \cos\big(\theta^2\big) \\ -\, m_2\, l \cos\big(\theta^2\big) & m_1 + m_2 \end{bmatrix} \tag{3–25}$$

3.3 Exercises

3.1 Derive the equation of motion of a mass particle in polar coordinates.

3.2 Derive the equation of motion of a mass particle moving on a smooth sphere.

3.3 Derive the Christoffel symbols $\left\{ {}_{k}{}^{j}{}_{l} \right\}_M$ for the system in Example 3.1.

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